Attitude Control of the Asteroid Redirect Robotic Mission Spacecraft with a Captured Boulder

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NASA’s Asteroid Redirect Robotic Mission (ARRM) aims to pick up a boulder from of a large asteroid and transport it to a distant retrograde orbit around the Moon for future exploration by a manned mission. In this paper, we present a detailed analysis for one of the main control challenges in ARRM, i.e., three-axis attitude control of the ARRM spacecraft with the captured boulder in the presence of large uncertainties in the physical model of the boulder. We first present a 30 degree-of-freedom nonlinear dynamic model of the ARRM spacecraft and boulder combination. We then linearize this nonlinear model about the nominal operating conditions to study the system’s modal properties. A finite element model of the ARRM spacecraft and boulder combination is used to validate our model. We then present linear and nonlinear control laws for the attitude control problem. Both the proportional-derivative based linear controller with lead-lag compensator and roll-off filter and the robust nonlinear tracking control law that tracks a derivative plus proportional-derivative based desired attitude trajectory give robust performance over the range of boulder parameters. We present a detailed comparison of these control laws and also present some design guidelines for the ARRM spacecraft.

I. Introduction

Multiple space agencies have announced plans for future small body exploration missions to near-Earth asteroids.1–3 National Aeronautics and Space Administration’s (NASA) Asteroid Redirect Robotic Mission (ARRM) is a robotic mission that aims to visit a large near-Earth asteroid, capture a multi-ton boulder from its surface, and redirect it into a stable distant retrograde orbit around the Moon. Subsequently, a manned mission will visit the asteroid to collect samples, which will enhance our knowledge of the formation of the solar system and the origin of life on Earth.4 The mission will also help demonstrate technologies for planetary defense and manned missions to Mars.5

A number of near-Earth asteroids have been identified as suitable candidates for the ARRM mission.6 Upon arrival at the target asteroid, the ARRM spacecraft will perform several fly-bys to scan the surface of the asteroid for suitable boulders that can be grasped by its robotic grippers. Due to the large uncertainties in the physical parameters for the asteroid and the boulder, the ARRM spacecraft should be capable of handling a wide range of boulder mass, size, shape, and inertia tensors. The conceptual ARRM spacecraft with the captured boulder is shown in Fig. 1.

In this paper, we focus on one of the main control challenges of ARRM, i.e., three-axis attitude control of the ARRM spacecraft with the captured boulder, that has uncertain physical parameters such as shape, size, mass and inertia. The schematic for the ARRM spacecraft is shown in Fig. 2. The ARRM spacecraft

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Figure 1: Artist’s rendering of the conceptual ARRM spacecraft (a) capturing a boulder form the asteroid surface, and (b) in transit with the captured boulder.

Figure 2: Schematic of the ARRM spacecraft

A. Problem Statement

This paper deals with the attitude control problem of the ARRM spacecraft with the captured boulder, after the ARRM spacecraft has taken off from the asteroid. The ARRM spacecraft and boulder combination should maintain a desired attitude during this stage of the mission, so that the propulsion system, solar panels, communication antennae and on-board sensors can be oriented in their appropriate directions. Moreover, uncontrolled spinning or tumbling of the spacecraft can excite structural vibrations, due to the large mass of the boulder, which might damage the spacecraft or lead to failure of the mission. Therefore, in this paper we develop an attitude controller capable of maintaining desired attitude of the ARRM spacecraft, in the
presence of large uncertainties in the boulder shape, size, mass and inertia tensor. Our attitude control law is capable of handling actuator saturation constraints and time delays in the estimation loop. To evaluate the control law’s performance using simulations, the following two scenarios were primarily used in this paper: (i) The spacecraft and boulder combination experiences a constant disturbance torque for an extended time, then the controller is turned on to bring it back to the desired attitude. (ii) The control law is used to slew the spacecraft and boulder combination from an initial attitude to a new desired final attitude.

B. Literature Survey

The attitude control problem investigated in this paper also arises in other space applications, like capture and removal of space debris and reviving or mining obsolete satellites in space. Therefore, the attitude control of a spacecraft with large uncertainties, due to the uncertainties in the captured object’s mass, inertia and physical properties is a topic of intense research. Nonlinear adaptive control, sliding mode control and robust $H_\infty$ linear control techniques have been proposed for this problem. In our previous work, we show that control laws that use feedback-linearization are unsuitable for this mission because of the large resultant disturbance torques created by the uncertain physical parameters. In Ref. 18, we propose a robust nonlinear tracking control law which stabilizes the combined system by robustly tracking a desired attitude trajectory that minimizes the resultant disturbance torques. The main limitation of these existing control techniques is that their application to the multiple degree of freedom ARRM spacecraft and asteroid combination system has not been investigated, because they are usually derived using simple rigid body dynamics. Therefore, in this paper, we compare the performance of linear and nonlinear controllers for this attitude control problem.

C. Main Contributions

The main contributions of this paper are as follows. Using the schematic of the ARRM spacecraft with the captured boulder in Fig. 2, two separate non-linear models are derived using different approaches and software packages: (i) the Euler–Lagrange nonlinear model is derived using the Matlab software, and (ii) the nonlinear model using Kane’s formulation is derived using the SDFAST software. These nonlinear models are simulated using Matlab Simulink, along with the control law, the sensors with measurement uncertainties, the actuators with saturation constraints and the time-delay in the estimation loop. These nonlinear models are also validated by comparing their open-loop simulations and the eigenvalues of their linear models with the finite-element-model derived at the Goddard Space Flight Center.

We then present linear and nonlinear control laws for the attitude control problem. The proportional-derivative based linear control law, with a lead-lag compensator and a roll-off filter, is designed using standard design techniques in linear control theory. The robust nonlinear tracking control law, which tracks a derivative plus proportional-derivative based desired attitude trajectory, guarantees robustness in the sense of finite gain $L_p$ stability for the nonlinear attitude control problem. The performance of these control laws is numerically evaluated for a range of asteroid parameters. Using theses simulation studies, we present procedures to design and tune the attitude controllers for the ARRM spacecraft.

II. Dynamics model of ARRM spacecraft

A. Nonlinear Dynamics Model

This model was developed using standard robotic modeling techniques and the Euler–Lagrange method. The derivation of the model was carried out using Matlab, heavily relying on the Symbolic Math toolbox.

1. Reference Frames

The system consists of several reference frames, one inertial frame and one local frame for each rigid body. The inertial frame is denoted by $I$. The body attached frames for the bus, +Y solar panel, -Y solar panel, +Y manipulator tip, -Y manipulator tip and boulder are denoted by $B, P, N, Q, V, A$ respectively. $B, P, N$ and $A$ are located at the center of mass of the respective bodies. The axes definition for the bus frame is depicted in Fig. 2. $Q$ and $V$ are located at the point of contact between the respective manipulator tip and the boulder. There are also several intermediate frames that establish kinematic chains for the solar
panel and manipulator joints. All of these intermediate frames were defined using the Denavit-Hartenberg convention,\textsuperscript{23} widely used for kinematic analysis of robotic manipulators. The use of this convention allows for automation of kinematic derivations and calculations in code.

Each solar panel arm is connected to the bus through a 3-DOF spherical joint. This joint is composed of three successive 1-DOF revolute joints along orthogonal axes. The intermediate frames for these kinematic chains can be seen in Fig. 3(a). Intermediate frame 1 is related to the bus frame $\mathcal{B}$ through a fixed rotation and translation. Frame 2 is obtained after a rotation of angle $\theta_{P_x}$ about the $z_1$ axis. Similarly, frame 3 is obtained after a rotation of angle $\theta_{P_y}$ about the $z_2$ axis. And, $\mathcal{P}$ is obtained after a rotation of angle $\theta_{P_z}$ about the $z_3$ axis and a fixed translation along the $y_3$ axis. It is important to note that frames 1, 2 and 3 are located at the same point on the spacecraft bus and that $\mathcal{P}$ is located at a different point, which is the center of mass of the solar panel. By looking at Fig. 3(a), it is evident that this sequence of rotations corresponds to a X-Y-Z Euler angles description of solar panel deflections with respect to the bus. Similar frames are defined for the -Y solar panel.

In the case of the robotic manipulators, the shoulder, elbow and wrist joints are 2-DOF joints capable of rotations about the $x_B$ and $z_B$ axes. This is equivalent to ignoring the torsional deformation of the manipulator links, which is justified due the high torsional stiffness afforded by short lengths and diameters of the rod-like links. The tip joint, which is the contact point between the manipulator and asteroid, is 3-DOF. Both of the manipulators are modeled through kinematic chains of 1-DOF revolute joints, similar to the solar panels. The intermediate frames for the +Y manipulator are depicted in Fig. 3(b).

These frame assignments lead to a 60 dimensional state space for the system. There are 6 coordinates that define the position and orientation of the bus wrt inertial frame; 3 each for the orientation of the solar panels wrt bus; and 9 each for the manipulators. These 30 position coordinates combined with corresponding velocity coordinates lead to the 60-dimensional system. However, the system has 6 constraints to ensure that both robotic manipulators are grasping the boulder properly. Thus, the system loses 6 degrees of freedom, resulting in a 24-DOF system.

Figure 3: Denavit-Hartenberg frames for (a) +Y solar panel, and (b) +Y manipulator

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The $30 \times 1$ generalized coordinates vector for the system is defined as

\[
q = [r_B, \theta_B, \theta_P, \theta_N, \theta_Q, \theta_V]^T
\]

\[
\theta_Q = [\theta_1, \theta_2, \theta_3, \theta_4]^T
\]

\[
\theta_V = [\theta_5, \theta_6, \theta_7, \theta_8]^T
\]

where, $\theta_P$ and $\theta_N$ are 3D solar panel X-Y-Z Euler angles. $\theta_1, \theta_2, \theta_3$ and $\theta_5, \theta_6, \theta_7, \theta_8$ are the 2D X-Z Euler angles for the respective manipulator joints as shown in Fig. 2. $\theta_4$ and $\theta_8$ are the 3D X-Y-Z Euler angles for the manipulator tip joints.

2. Kinematics

Position Kinematics

For the Bus, the position and orientation of the center of mass are denoted by position vector and roll-pitch-yaw Euler angles, $r_B, \theta_B$, expressed in the inertial frame. All other vectors are expressed in the bus frame, unless stated otherwise. For vectors not in the bus frame, the frames are expressed through a left superscript.

Deriving kinematic relations within the system is simplified by the use of the D-H convention, which allows for easy derivation of the homogenous transformation matrices between frames of a kinematic chain. A homogenous transformation matrix is a $4 \times 4$ matrix that encodes both relative rotations and translations between frames.\(^{24}\) These transforms have the following structure.

\[
T_{B}^{P} = \begin{bmatrix}
R & d \\
0 & 1
\end{bmatrix}
\]  

(1)

where, $T_{B}^{P}$ transforms a vector from $P$ to $B$, $R$ is an orthonormal rotation matrix and $d$ is a 3D vector.

To use homogenous transforms, 3D position vectors are converted to 4D homogenous position vectors, simply by appending 1 to them. For instance the origin in homogenous coordinates is written as $\vec{0} = [0, 0, 0, 1]^T$.

Thus, if $r_P$ denotes the position of the center of mass of the +Y solar panel expressed in the bus frame, then

\[
r_P = T_{B}^{P}(\theta_P)\vec{0}
\]

(2)

Note that $T_{B}^{P}$ is a function of the angles $\theta_P$.

The other positions can be represented as

\[
r_N = T_{N}^{A}(\theta_N)\vec{0}
\]

(3)

\[
r_{AQ} = T_{AQ}^{A}(\theta_Q)T_{AQ}^{B}\vec{0}
\]

(4)

\[
r_{AV} = T_{AV}^{B}(\theta_V)T_{AV}^{Q}\vec{0}
\]

(5)

where the boulder position is calculated through both the +Y and -Y manipulators. The transformations $T_{AQ}^{A}$ and $T_{AQ}^{B}$ are fixed as they only depend on the radius of the boulder, $R_A$ and angle $\phi$, as shown in Fig. 2, both of which are fixed quantities. In the interest of brevity, from here on, the kinematic expressions for each rigid body in the system will not be shown since they all follow the same structure. The expressions can be found in the Appendix.

For the links on the manipulator arms, the intermediate frames in the D-H assignment can be used to calculate the positions of the +Y manipulator links’ center of masses (SE1, EW1, WT1) and the -Y manipulator links’ center of masses (SE2, EW2, WT2). The lengths of the manipulator links are denoted by $l_{SE}, l_{EW}, l_{WT}$. Thus, in the bus frame, the center of masses are

\[
r_{SE1} = T_{SE1}^{B}[-\frac{l_{SE}}{2}, 0, 0, 1]^T
\]

(6)

Here, $T_{SE1}^{B}$ is a function of $\theta_1$. 
Velocity Kinematics

The linear and angular velocities of the bus are denoted by \( v_B, \omega_B \). When expressed in the bus frame, these are related to the generalized coordinates derivatives in the following manner.

\[
v_B = R_B^I \dot{r}_B \\
\omega_B = J_{\omega_B} \dot{\theta}_B
\]

where, \( R_B^I \) is the \( 3 \times 3 \) rotation matrix from the inertial frame to the bus frame, and \( J_{\omega_B} \) is a \( 3 \times 3 \) matrix that transforms the Euler angle derivatives to angular velocity.

The velocity kinematics for the rest of the system can be derived easily. The D-H convention allows for a formulaic calculation of the Jacobian matrices for kinematic chains. A Jacobian matrix is a transformation that transforms the Euler angle derivatives to angular velocity.

To calculate the velocities with respect to the inertial frame, the rotation and translation of the bus need to be accounted for. The linear and angular velocities of the \(+Y\) solar panel with respect to the inertial frame, expressed in the bus frame are

\[
v_p = v_B - S(r_P)\omega_B + v_p^B \\
\omega_p = \omega_B + \omega_p^B
\]

where, \( S(r) \) is the skew-symmetric, cross product matrix for the vector \( r \). These can be rewritten as

\[
v_p = R_B^I \dot{r}_B - S(r_P)J_{\omega_B} \dot{\theta}_B + J_{vp} \dot{\theta}_P \\
\omega_p = J_{\omega_B} \dot{\theta}_B + J_{\omega_p} \dot{\theta}_P
\]

Note that \( r_P \) can be calculated given the current system state, using Equation 2.

Similar equations are derived for frames \( N, Q, V \) and \( A \). Using these equations, all the velocities can be written as the product of a \( 3 \times 30 \) matrix and the derivatives of the generalized coordinates.

\[
v_B = [R_B^I, 0, 0, 0, 0, 0] \dot{\mathbf{q}} = J_B^I \dot{\mathbf{q}} \\
v_p = [R_B^I, -S(r_P)J_{\omega_B}, J_{vp}, 0, 0, 0] \dot{\mathbf{q}} = J_P^I \dot{\mathbf{q}} \\
v_{SE1} = [R_B^I, -S(r_{SE1})J_{\omega_B}, 0, 0, J_{v_{SE1}}, 0] \dot{\mathbf{q}} = J_{v_{SE1}}^I \dot{\mathbf{q}} \\
v_{AQ} = [R_B^I, -S(r_{AQ})J_{\omega_B}, 0, 0, J_{v_{AQ}}, 0] \dot{\mathbf{q}} = J_{v_{AQ}}^I \dot{\mathbf{q}}
\]

\[
\dot{\mathbf{q}} = [\dot{r}_B, \dot{\theta}_B, \dot{\theta}_P, \dot{\theta}_N, \dot{\theta}_Q, \dot{\theta}_V]^T
\]

Note that in formulating the kinematics of the boulder frame \( A \), the position and velocity of the boulder are calculated using both the \(+Y\) and \(-Y\) manipulators. The motivation for this will become apparent when formulating the dynamics of the problem.

The same treatment is applied for the angular velocities.

\[
\omega_B = [0, J_{\omega_B}, 0, 0, 0] \dot{\mathbf{q}} = J_B^I \dot{\mathbf{q}} \\
\omega_p = [0, J_{\omega_B}, J_{\omega_P}, 0, 0] \dot{\mathbf{q}} = J_P^I \dot{\mathbf{q}} \\
\omega_{SE1} = [0, J_{\omega_B}, 0, 0, J_{\omega_{SE1}}, 0] \dot{\mathbf{q}} = J_{\omega_{SE1}}^I \dot{\mathbf{q}} \\
\omega_{AQ} = [0, J_{\omega_B}, 0, 0, J_{\omega_{AQ}}, 0] \dot{\mathbf{q}} = J_{\omega_{AQ}}^I \dot{\mathbf{q}}
\]
3. Euler–Lagrange Formulation

Deriving the dynamic equations governing the system via the Euler–Lagrange method requires a complete description of all kinds of energies associated with the system. In our system, energy is gained or lost in the form of: kinetic energy possessed by the rigid bodies; elastic potential energy due to the springs in the joints; energy dissipated at the viscous dampers in the joints; and generalized torques produced by the bus mounted thrusters. The effects of any gravitational fields are neglected, since these are insignificant in deep space.

Kinetic Energy

The kinetic energy of the system comprises of the translational and rotational kinetic energies for every rigid body. The masses of these bodies are represented by \( m_B, m_P, m_N, m_A, m_{SE}, m_{EW} \) and \( m_W \) and the inertia tensors are represented by \( I_B, I_P, I_N, I_A, I_{SE}, I_{EW} \) and \( I_{WT} \). All the inertia tensors are expressed in their respective body frames and are consequently, fixed quantities.

With this nomenclature, the kinetic energy of the bus can be expressed as

\[
T_B = \frac{1}{2} v_B^T m_B v_B + \frac{1}{2} w_B^T I_B w_B
\]

And using Eqs. (55) and (66), this can be rewritten in terms of the generalized coordinates as

\[
T_B = \frac{1}{2} q^T [J_{v_B}^T m_B J_{v_B}^T + J_{\omega_B}^T I_B J_{\omega_B}^T] \dot{q}
\]

(21)

Note that the terms within the brackets represent a 30 \( \times \) 30 matrix.

To express the kinetic energy of the +Y solar panel, its angular velocity has to be transformed to the \( P \) frame, wherein the inertia tensor of the panel is expressed.

\[
T_P = \frac{1}{2} v_P^T m_P v_P + \frac{1}{2} (R_B^p w_P)^T I_P (R_B^p w_P)
\]

And using Eqs. (56) and (67), this is rewritten in terms of the generalized coordinates.

\[
T_P = \frac{1}{2} \dot{q}^T [J_{v_P}^T m_P J_{v_P}^T + (R_B^p I_{\omega_P}^I)^T I_P (R_B^p I_{\omega_P}^I)] \dot{q}
\]

(22)

The same approach is followed to derive expressions for the kinetic energies of other bodies in the system.

\[
T_{SE1} = \frac{1}{2} q^T [J_{v_{SE1}}^T m_{SE} J_{v_{SE1}}^T + (R_B^{SE1} I_{\omega_{SE1}}^I)^T I_{SE} (R_B^{SE1} I_{\omega_{SE1}}^I)] \dot{q}
\]

\[
T_{AQ} = \frac{1}{2} q^T [J_{v_{AQ}}^T m_{AQ} J_{v_{AQ}}^T + (R_B^{AQ} I_{\omega_{AQ}}^I)^T I_{AQ} (R_B^{AQ} I_{\omega_{AQ}}^I)] \dot{q}
\]

(23)

(24)

To calculate the kinetic energy of the boulder, it is split into two bodies, as is common while modeling systems with closed topologies.\(^{25-28}\) Each of the split boulders has the same shape and size as the original but only half the density. It is assumed that one of the split boulders is attached to the +Y manipulator tip and the other to the -Y manipulator tip. Constraints are added to the dynamics to ensure that these split halves are always superimposed, resulting in the original boulder. These constraints are discussed in a later section.

The kinetic energy for the system is the sum of the kinetic energies of all rigid bodies.

\[
T = T_B + T_P + T_N + T_{AQ} + T_{AV} + T_{SE1} + T_{EW1} + T_{WT1} + T_{SE2} + T_{EW2} + T_{WT2}
\]

In taking this sum, all the terms within the brackets can be combined into a single 30 \( \times \) 30 matrix, \( M \). This is the mass matrix of the system and encodes the generalized mass properties of the system. The mass matrix of any physical system is symmetric, positive definite and invertible.

\[
T = \frac{1}{2} \dot{q}^T M \dot{q}
\]

(25)
Potential Energy

Every joint within the spacecraft has an attached rotational spring and a viscous damper, in parallel with each other. The springs contribute to the potential energy for the system, while the dampers contribute to the dissipation energy discussed next.

Each joint has a nominal angle about which the spring acts. When deflected from these nominal angles, the springs produce a restoring torque about the axis of the joint. These nominal angles are trivial for the solar panel joints. However, for the manipulator joints, these angles depend upon the configuration in which the boulder is grasped. Since there are infinite configurations for grasping a boulder of a given size and shape, some sort of pre-selection needs to be performed in order to determine the nominal angles. To this end, a pose function is chosen to define the grasping configuration. Since the model assumes a spherical boulder, the pose function depends only on the radius of boulder. Some features of the pose function are:

- Both manipulators lie completely in the Y-Z plane of the bus frame.
- The manipulator links mirror each other across the X-Z plane of the bus frame.
- The point of contact between the manipulator tips and the boulder are symmetrical about the X-Z plane of the bus.
- These points are defined by a pre-selected, fixed angle, $\phi$.
- The wrist-tip links of the manipulators are perpendicular to the boulder surface.

Once the pose angles are defined, the potential energy can be written as

$$V = \frac{1}{2} (q - q_{\text{pose}})^T K (q - q_{\text{pose}})$$  \hspace{1cm} (26)

where $K$ is a $30 \times 30$ diagonal matrix with the diagonal entries corresponding to the spring constants, $k$ for the respective joints.

Rayleigh Dissipation Energy

The Rayleigh dissipation function captures the energy lost through the viscous dampers at the joints. It is mathematically similar to the potential energy function.

$$R = \frac{1}{2} \dot{q}^T C \dot{q}$$  \hspace{1cm} (27)

where $C$ is a $30 \times 30$ diagonal matrix with the diagonal entries corresponding to the damping constant, $c$ for the respective joints.

Generalized Forces

The generalized forces represent the energy introduced in a system by external forces and moments. In this system, the only external moments are those generated by the three orthogonal thrusters mounted on the bus. To convert these torques, $\tau_B$, into generalized torques for the Euler-Lagrange framework, they are simply left multiplied by the transposed Jacobian matrix for the angular velocity of the bus.

$$Q = [0, J_{\omega_B}^T \tau_B, 0, 0, 0, 0]^T$$  \hspace{1cm} (28)

$Q$ represents the $30 \times 1$ vector of generalized forces acting on each coordinate of the system.

4. Euler Lagrange Dynamics

With the system energetics determined, deriving the dynamic equations involves defining the Lagrangian for the system.

$$L = T(q, \dot{q}) - V(q)$$

and the dynamics are given by the Euler-Lagrange equations for each coordinate, $q_i$

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} + \frac{\partial R}{\partial \dot{q}_i} - Q_i = 0$$  \hspace{1cm} (29)
This presents a set of 30 governing equations, one from each generalized coordinate. After some algebraic manipulation, these equations can be written in matrix form as

\[ M(q)\ddot{q} + C(q, \dot{q})\dot{q} + \phi(q) + \nabla R(\dot{q}) - Q = 0 \]  

(30)

Here, \( M(q) \) is the same Mass matrix defined before. \( C(q, \dot{q}) \) is called the Coriolis matrix and its elements are defined as follows

\[ C_{k,j}(q, \dot{q}) = \sum_{i=1}^{30} \frac{1}{2} \left( \frac{\partial M_{kji}}{\partial q_i} + \frac{\partial M_{kij}}{\partial q_j} - \frac{\partial M_{kji}}{\partial q_k} \right) \dot{q}_i \quad \text{for } k, j = 1, 2, ... 30 \]  

(31)

In this definition, the elements are defined using Christoffel symbols. \( \phi(q) \) is the gradient of the potential energy with respect to \( q \)

\[ \phi(q) = \nabla V(q) \]  

(32)

\( \nabla R \) is the gradient of the Rayleigh dissipation function with respect to \( \dot{q} \). Equation (30) represents the unconstrained dynamics for the system, wherein the two split boulders are not overlapping each other and can move independently.

5. Split Boulder Constraint

Definition

As explained previously, the modeling involves splitting the boulder into two boulders of the same shape and size as the original but with half the density. To ensure that both split boulders superimpose each other, resulting in the original boulder, constraints need to be defined on the manipulator angles. To do so, a total of six constraint equations are required: three equations ensure that the centers of both split boulders coincide; and three equations ensure that the relative orientation of both split boulders with respect to the bus, is the same.

The position constraints relates the center of masses of both split boulders. The position of the origin of the boulder frame is calculated through both manipulators and then equated.

\[ ^B r_A = ^T_Q T_A^B \vec{0} = ^T_B T_A^Y \vec{0} \]

This constraint is written in the following form

\[ h_1(q) = [T_Q^B T_A^B - T_B^B T_A^Y] \vec{0} = 0 \]

Note that we only use the first three rows from this constraint equation. The fourth row exists because of the use of homogenous coordinates and is meaningless \((1 - 1 = 0)\).

The orientation constraint is enforced by equating the \([1, 1, 1]^T\) vector from both split boulder frames.

\[ h_2(q) = [R_Q^B R_A^Y - R_Q^B R_A^Q][1, 1, 1]^T = 0 \]

Alternatively, the three orientation constraints can also be written using a quaternion representation of the boulder orientation wrt bus. To do so, the rotation matrices, \( R_Q^{AQ} \) and \( R_Q^{AY} \) are converted to quaternions and then equated to get three constraining equations.

\[ (R_Q^{A_Q}(3, 2) - R_Q^{A_Q}(2, 3)) - (R_Q^{A_Y}(3, 2) - R_Q^{A_Y}(2, 3)) = 0 \]

\[ (R_Q^{A_Q}(1, 3) - R_Q^{A_Q}(3, 1)) - (R_Q^{A_Y}(1, 3) - R_Q^{A_Y}(3, 1)) = 0 \]

\[ (R_Q^{A_Q}(2, 1) - R_Q^{A_Q}(1, 2)) - (R_Q^{A_Y}(2, 1) - R_Q^{A_Y}(1, 2)) = 0 \]

Both approaches to defining the orientation constraints are identical in theory but differ slightly in practice, due to numerical considerations.

The position and orientation constraints are combined into the \(6 \times 1\) vector constraint \( h(q) \). This defines the constraints for the system.

\[ h(q) = \begin{bmatrix} h_1(q) \\ h_2(q) \end{bmatrix} = 0 \]  

(33)
This is a set of holonomic constraints, as they do not depend on the generalized coordinate rates. Holonomic constraints can be removed from the system dynamics by eliminating the redundant states, thereby reducing the dimensionality of the system. In this case, six of the manipulator joint angles are dependent states and can be removed from the state space by using the constraint equations. While it may look like this approach would simplify the dynamics, that is not necessarily the case. Since the joints have springs and dampers associated with them, any calculation of the system energies will require computing the current joint angles. This will be done by solving the highly nonlinear inverse kinematic equations for the dependent joint angles at every time step of the simulation. These equations are computationally expensive to solve and often have multiple solutions. This performance drop offsets the gains obtained by reducing the dimensionality of the system and eliminating the constraint equations.

Another factor to consider while modeling closed kinematic chains, is the choice of the point where the chains are joined together via the constraint. In the case of this model, this point is the center of mass of the boulder after it is split into two rigid bodies. The alternative is to close the chain at one of the joints. This approach leads to elimination of that joint from the state space and a reduction in the number of constraints required. However, it suffers from the same problem regarding calculation of system energies through solving inverse kinematic equations. Hence, this approach was avoided and the chain was closed by splitting a rigid body. Any of the manipulator links could have been split for this purpose, but the boulder was chosen in order to obtain the simplest set of kinematic equations. To achieve the simplest set of equations, the length of the kinematic chains needs to be kept at a minimum. Besides, splitting the boulder also allows for a uniform treatment of both manipulators.

**Enforcement**

The constrained system is represented through thirty ordinary differential equations (ODE) and six algebraic equations, comprising a set differential-algebraic equations (DAE). The simplest method to solve such DAE systems is to differentiate the algebraic equations twice, to express them at the acceleration level. Then, they can be augmented with the other ODEs resulting in a pure ODE system. This method of enforcing constraints performs rather erratically around configurations where one or more of the constraint equations is redundant. This is the case for the ARM system in the pose configuration. Since the manipulators lie in the Y-Z plane, one of the split boulder orientation constraint is redundant. Thus, to ensure accurate results from the constraint enforcement, the method proposed by Aghili\textsuperscript{31} is utilized. This method involves projecting the constrained dynamics onto a reduced space using a projection operator.\textsuperscript{32} This method deals with the presence of redundant constraints and also allows for faster simulations than some of the other approaches.

Aghili’s method involves projecting the dynamics onto a reduced space, defined by the algebraic constraint equations. To do so, a projection operator, $P$, is defined to orthogonally project any vector to the null space of a linear transformation.

For, $A : \mathbb{R}^n \rightarrow \mathbb{R}^m$

Define, $P \in \mathbb{R}^{n \times n}$

such that, $Pq \in \mathcal{N}(A) \quad \forall \quad q \in \mathbb{R}^n$

Finding such a $P$ matrix is straightforward using the singular value decomposition of $A$. However, it can also be calculated with the use of a pseudoinverse.

$$P = I - A^+ A$$

where, $A^+$ denotes the pseudoinverse of $A$. This definition of $P$ is more practical to use during simulations.

The constraints in the system are represented by Eq (33). These can be differentiated twice to obtain

\begin{align*}
A\dot{q} &= 0 \quad (35) \\
A\ddot{q} - A\dot{q} &= 0 \quad (36)
\end{align*}

where, $A = \partial h/\partial q$ is the Jacobian of the constraint vector wrt generalized coordinates. It is evident that the allowable velocities for the system belong to the null space of the constraint Jacobian. This is where the projection operator will be utilized to project the accelerations and velocities of the system to the allowable subspace defined by the constraint equations.
The constraints are added into the system with the use of an augmented Lagrangian using the Lagrange multipliers $\lambda$. These Lagrange multipliers generate a constraint force $\mathcal{F}$ based on the Jacobian matrix.

$$\mathcal{F} = A^\top \lambda$$

When this constraint force is added into the dynamics, the projection operator can be used to eliminate the constraint forces and Lagrange multipliers to obtain the following set of constrained dynamic equations of the system.\(^{31}\)

$$M_c(q)\ddot{q} = N_c(q, \dot{q})\dot{q} - [C(q, \dot{q})\dot{q} + \phi(q) + \nabla R(\dot{q}) - Q]$$ (37)

$$M_c(q) := M + PM - (PM)^\top$$ (38)

$$N_c(q, \dot{q}) := MN$$ (39)

In this set of equations, the $M_c$ matrix is referred to as the constraint inertia matrix. It can be shown to be invertible and positive definite. However, it lacks the symmetricity of the mass matrix.

### B. Linearized Model

The lumped mass Euler-Lagrange model can be linearized using Taylor series expansions in order to perform small perturbation analysis around the equilibrium configuration.\(^{33}\) The linearized model is expressed in state space and transfer function forms and frequency domain techniques are used to understand the behaviour of the system. Starting from the Eq. (39) for the constrained dynamics, each term is approximated by its first-order Taylor expansion around the equilibrium. The equilibrium point comprises of zero deflections for all states except the manipulator angles which are locked in the pose configuration to grasp the boulder. This equilibrium state is denoted by $q_0$. After linearization, the system is defined by the perturbation states $z = q - q_0$.

$$M_c(q)\ddot{z} = \left( M_c(q_0) + \left. \frac{\partial M_c}{\partial q} \right|_{q_0} z \right) \ddot{z}$$

$$N_c(q, \dot{q})\dot{z} = \left( N_c(q_0, \dot{q}_0) + \left. \frac{\partial N_c}{\partial q} \right|_{q_0, \dot{q}_0} z + \left. \frac{\partial N_c}{\partial \dot{q}} \right|_{q_0, \dot{q}_0} \dot{z} \right) \ddot{z}$$

$$C(q, \dot{q})\dot{z} = \left( C(q_0, \dot{q}_0) + \left. \frac{\partial C}{\partial q} \right|_{q_0, \dot{q}_0} z + \left. \frac{\partial C}{\partial \dot{q}} \right|_{q_0, \dot{q}_0} \dot{z} \right) \ddot{z}$$

$$\phi(q) = \phi(q_0) + \left. \frac{\partial \phi}{\partial q} \right|_{q_0} z$$

$$\nabla R(\dot{q}) = \nabla R(\dot{q}_0) + \left. \frac{\partial \nabla R}{\partial \dot{q}} \right|_{\dot{q}_0} \dot{z}$$

$$Q(q, \tau_B) = Q(q_0)\tau_B$$

After neglecting all the terms that are second order in the above expression

$$M_{c_0}\ddot{z} = N_{c_0}\dot{z} - C_0\dot{z} - \phi_0 - \left. \frac{\partial \phi}{\partial q} \right|_0 z - \nabla R_0 - \left. \frac{\partial \nabla R}{\partial \dot{q}} \right|_0 \dot{z} + Q_0(\tau)$$ (40)

where the subscript $*_0$ represents the evaluation of $*$ at the equilibrium point. By definition, $\phi_0 = 0$ and $\nabla R_0 = 0$. Also, at zero velocities, $C_0 = 0$. From Eqs. (26) & (27), $\left. \frac{\partial \phi}{\partial q} \right|_0 = K$ and $\left. \frac{\partial \nabla R}{\partial \dot{q}} \right|_0 = C$. This leads to the dynamics of the linearized system.

$$M_{c_0}\ddot{z} = (N_{c_0} - C)\dot{z} - Kz + Q_0(\tau)$$ (41)

This set of dynamic equations can be represented in the state space form with the following system matrix.

$$A = \begin{bmatrix} 0 & I \\ -K & (N_{c_0} - C) \end{bmatrix}$$ (42)
Figure 4: Bode plot comparison for the linearized Matlab (red), SDFAST (blue) and NASTRAN (green) models. (a) Torque about bus X axis to bus X angle (b) Torque about bus Y axis to bus Y angle (c) Torque about bus Z axis to bus Z angle
Figure 5: Lowest frequency structural mode for the plant

Table 1: ARRM Spacecraft Parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bus Mass (kg)</td>
<td>9246.8</td>
</tr>
<tr>
<td>Solar Array Mass (kg)</td>
<td>196.3</td>
</tr>
<tr>
<td>Shoulder-Elbow Link Mass (kg)</td>
<td>38.9</td>
</tr>
<tr>
<td>Elbow-Wrist Link Mass (kg)</td>
<td>38.9</td>
</tr>
<tr>
<td>Wrist-Tip Link Mass (kg)</td>
<td>30.3</td>
</tr>
<tr>
<td>Bus Half-Width, D1 (m)</td>
<td>1.5</td>
</tr>
<tr>
<td>Bus CM to Array Beams, D2 (m)</td>
<td>-2.59</td>
</tr>
<tr>
<td>Bus CM to Base, D3 (m)</td>
<td>3.3</td>
</tr>
<tr>
<td>Bus CM to Solar Array Hinge, D4 (m)</td>
<td>3.5</td>
</tr>
<tr>
<td>Solar Array Hinge to Solar Array CM, D5 (m)</td>
<td>4.8</td>
</tr>
<tr>
<td>Boulder Grab Angle, $\phi$ (deg)</td>
<td>-30</td>
</tr>
</tbody>
</table>

Table 2: Boulder Parameters

<table>
<thead>
<tr>
<th>Boulder Diameter (m)</th>
<th>0</th>
<th>2</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Boulder Mass (tons)</td>
<td>0</td>
<td>8.7</td>
<td>70.0</td>
<td>136.7</td>
</tr>
</tbody>
</table>
The natural frequencies and modeshapes for this linear model are examined and compared with those from the linearized NASTRAN FEM model. Furthermore, the SDFAST nonlinear model is also linearized using numerical linearization techniques and the Linear Analysis functionality of the Control System Toolbox in Simulink. This allows for comparison of all three models: Matlab, SDFAST and NASTRAN. Figure 4 shows the SISO Bode plots for these models from torque to rotation about an axis. This knowledge of modeshapes is used to inform the control design as shown later. The analysis is also very important for understanding the challenges in control design. For instance, the lowest frequency mode, shown if Fig. 5 constrains the bandwidth of the system because excitation of this mode could lead to instability if the controller is not carefully designed.

From the analysis, it is evident that the SDFAST model captures the dominant modes seen in the NASTRAN model more accurately than the Matlab model. The differences observed between the MATLAB and SDFAST models may be attributed to the difference in constraint enforcement methodologies and the differing propagation of numerical errors through the different modeling approaches. The Matlab model uses fixed precision representations for all numerical substitutions of symbolic variables. This is required to keep the physical size of the equations manageable. Because of the inefficiency of algebraic manipulation and differentiation of the symbolic expressions, the resulting expressions can reach several gigabytes of text data if the numerical precision is not compromised. Such large files are highly undesirable because they lead to very slow simulations. Thus, the numerical precision was compromised. On the other hand, the SDFAST software package has been designed and optimized primarily to produce models for greater numerical accuracy and faster simulations. Kane’s formulation does not involve taking derivatives of the kinematics to formulate the acceleration level dynamics; instead it uses cross products to achieve the end result. Consequently, the propagation of numerical errors is limited in this formulation. To achieve this accuracy and speed however, SDFAST compromises on equation structure and readability in the form of Eq. (39), which is desirable for control design and analysis.

The model parameters used for the control design, simulation and analysis are listed in Tables 1 and 2. All models have spherical boulders, with the nominal boulder diameter being 4 m and mass 70 tons. Non-nominal cases of boulder diameters of 0 m, 2 m and 5 m are also considered.

III. Linear and Nonlinear Control Synthesis for ARRM

The control problem here is to regulate the attitude of the spacecraft using the onboard thrusters. In that sense, the system has 6 output states and 3 input controls. It also has a 3/64 s actuation time delay which further complicates the control design because of the high frequency phase roll-off. For this problem, a linear and a nonlinear controller were designed and compared using generalized concepts of linear, frequency domain robustness and performance metrics.

- **Bandwidth:** For SISO linear systems, bandwidth may be calculated through the use of Bode plots. However, for nonlinear systems that are not possible. In our analysis the bandwidth is calculated by simulating the system response to sinusoidal inputs of varying frequencies. Bandwidth is defined as the minimum frequency where the system response to sinusoidal excitation has an amplitude greater than 70% of the input signal.

- **Generalized Gain Margin:** Again, gain margins for nonlinear systems cannot be computed through Bode plots. The Generalized Gain Margin is calculated by adding a variable gain in the system after the controller stage. This gain value is ramped up until the first occurrence of instability in the system response. That value of gain is considered the GGM for the controller.

- **Generalized Phase Margin:** Similarly, the Generalized Phase Margin is calculated by adding a variable time delay in the system after the controller stage and ramping it up until the first occurrence of instability in the system response. That value of time delay (in seconds) is considered the GPM for the controller.

- **Flex Margin:** Since the system has uncertain modes that can move around slightly in the Bode plots, it does not make sense to calculate robustness margins as done classically. For instance, a large uncertain mode near the phase crossover point used to calculate gain margin could move around a little bit and significantly cut down on the margin. Thus, the concept of flex margin is used. Flex margin is
defined as the minimum possible gain margin in the system in a region of ±30° from the phase crossover frequency.

To validate the control algorithm and to analyze the system behavior, extensive simulations were run. In particular, two scenarios are considered:

- Scenario 1: A 200 Nm disturbance torque is applied across the X & Y axis for 10s. Then the controller is switched on to stabilize the system.
- Scenario 2: From nominal state, the system is slewed to 10 deg along an axis.

The following sections detail the control design process and shows results of simulations for these two scenarios.

A. Linear Control Law

For the attitude control of the ARRM spacecraft, a linear controller was designed to accomplish multiple, essential tasks. A linear control law offers slower dynamic response compared to a nonlinear control law. However, a linear controller offers various advantages, like architectural simplicity and minor computational costs. Thus, the linear controller represents a weighted trade-off between less ambitious performance but a more economical solution. The linear controller will be compared with the nonlinear controller in Sec. III C.

The design procedure pursued for the linear controller is a frequency domain approach with a synthetic method for the loop-shaping of the controller.34–37 Through the use of open-loop Bode plots of the linearized system dynamics, the frequency responses are divided into three different intervals based on the bandwidth and crossover frequencies: low-frequency \((−\infty, \omega_b]\), mid-frequency \((\omega_b, \omega_c]\), high-frequency \([\omega_c, \infty]\). For each interval, a relative sub-controller is built and optimized to satisfy the desired requirements.

In the low frequency interval, the gain should be high, to minimize the tracking reference error and to move \(\omega_b\) to a higher frequency. The presence of the first structural mode in this region constrains the position of \(\omega_b\) as evidenced by its response speed. Since the nature of the model is intrinsically uncertain, the usage of structural filters to remove or decrease the amplitude of this structural mode is ineffective. As a result, these filters run efficiently, but only locally in the frequency in which it is located.

In the high frequency interval, the gain should be low, for adequate noise attenuation and the presence of uncertain structural modes requires the lowest possible gain amplitude. At high frequencies the response became sensitive to the variability of the structure properties, thus same material exposed to different manufactures could present very different answers. Moreover, the computational cost for the FEM model creation increases broadly for shorter wavelengths and for higher frequencies, because of the larger number of necessary details for the creation.

In the mid frequency interval, large robustness margins are required to maintain system stability in presence of all the structural modes.

<table>
<thead>
<tr>
<th>Table 3: Linear Controller Design Requirements</th>
</tr>
</thead>
<tbody>
<tr>
<td>Phase Margin</td>
</tr>
<tr>
<td>---------------</td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td>Gain Margin</td>
</tr>
<tr>
<td>Flex Gain Margin</td>
</tr>
</tbody>
</table>

Table 3 shows design specifications for the robustness margins with the linear controller. The Flex Gain Margin represents the minimum gain value in a phase interval of ±30 deg around the frequency point where the phase is ±180 deg.

The controller was tuned using the FEM model and verified on the lumped mass models. Three different FEM models are taken into account to consider the captured boulder uncertainty. These differ from each other in the boulder size, mass and inertia. For the nominal case, the boulder diameter is 4 meters, while for the non-nominal cases it is 2 meters and 0 meters respectively.
The analyzed system is a MIMO with 3 torque inputs, 3 exogenous inputs, 3 sensed outputs, which are the rotational angles of the bus center of mass with respect to the inertial reference frame, and 15 regulated outputs. The cross-coupled transfer functions, such as torque about x-axis to rotation about y-axis, are neglected as the amplitude peaks of their structural modes are low compared to those for the direct transfer functions. Thus, the analysis is carried out with only 3 SISO systems.

To be economical, the architectures of the controller algorithms for each axis are the same but they are all tuned independently. These have been optimized with MATLAB’s SISO Design Tool GUI. To validate and test the controllers, extensive computer simulations are performed with several challenging scenarios.

1. Linear Controller Design

The Bode plots for the open loop plant dynamics and closed loop plant with the designed controller for the three boulder size cases are depicted in Fig. 6. The first structural mode in the y-axis plot is highlighted in red. This is the limiting factor for the response speed of the controller.

![Bode plots of the FEM model](image)

Figure 6: Bode plots of the FEM model.

After several iterations, the following control architecture was chosen.

\[
G_c = K_p \left(1 + \frac{K_d}{K_p} s\right) \frac{1 + T_2 s}{1 + \beta T_2 s} \frac{1 + T_1 s}{1 + \alpha T_1 s}^2
\]  

Equation (43) is composed by a PD controller, a lead controller, a first order Roll-off Filter, and a lag controller.

**PD:** The direct measurement of angular rates of the spacecraft through gyroscopic sensors allows for the usage of a pure PD term in the controller algorithm. The proportional gain of the controller increases the DC gain, while the derivative gain moves the intersection of the transfer function with 0dB line towards a higher frequency and increases the phase in the proximity of this intersection. Therefore, the PD term provides a higher bandwidth. The crossing frequency is increased to the limit imposed by the robustness requirements.

**Lag:** The Lag term \((\beta > 1)\) increases the phase in the mid-frequency interval by changing \(T_2\), where the majority of structural modes are located. Thus, the system has adequate phase at every 0dB crossing in this region.
**Lead - Roll-off:** The third term is a combination of a Lead controller ($\alpha < 1$) and a first order Roll-off filter with the poles for each at the same location, hence the squared denominator. Both of these work in the same frequency interval but for different purposes. The $T_1$ term in the Lead is used to shift the first phase intersection with the -180 degree line to a higher frequency, so that this intersection occurs after the high-gain structural modes in this region. Moreover, the roll-off filter reduces the gain of the highest structural modes. The utilization of narrow-band filters that locally affect structural modes, such as notch filters, is not possible due to the uncertainty in the system.

This designed controller satisfies all the design specifications and allows for a bandwidth of 0.02 Hz. All of the finalized parameters of the designed controller are tabulated in the Table 4 and the Closed Loop bandwidths are visible in the Bode plots of Fig. 7.

Table 4: Linear control margins for the three boulder mass cases for each diagonal SISO term in the MIMO matrix

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Diagonal Term 1-1</th>
<th>Diagonal Term 2-2</th>
<th>Diagonal Term 3-3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega_b$ [Hz]</td>
<td>0.023 0.04 0.08</td>
<td>0.02 0.045 0.16</td>
<td>0.02 0.04 0.065</td>
</tr>
<tr>
<td>$PM_{min}$ [deg]</td>
<td>77.2 62.9 50.5</td>
<td>61.1 56.7 49.9</td>
<td>62.1 58.4 59.1</td>
</tr>
<tr>
<td>$GM_{min}$ [dB]</td>
<td>30.4 20 30</td>
<td>23.9 22.3 23</td>
<td>20.9 32.7 31.2</td>
</tr>
<tr>
<td>$FM_{min}$ [dB]</td>
<td>27.1 19.1 19.2</td>
<td>19.1 18.7 20.5</td>
<td>30.1 31 29.6</td>
</tr>
</tbody>
</table>

Figure 7: Closed loop Bode plot of FEM model with linear controller.

The Nichols charts for all diagonal SISO terms in the MIMO matrix for the three boulder mass cases are reported in Fig. 8. In the Nichols charts, all of the crossings with the $\pm 180 + 360n$ lines and the corresponding Flexible Margins (FM) are visible.

2. **Linear Controller Simulations with Linear Plant**

This section presents the results of the simulations with the linearized FEM model for the two scenarios discussed earlier. In the Comparison of Control Law section, the same simulations are run with the nonlinear system model.

Figures 9, 10, 11, and 12 show the system response and control effort of the linear controller for both scenarios. However, the approach of this process is conservative. In fact, the dynamic equations are being linearized for specific conditions around a single linearization point that involves a specific pose for the manipulator arms and the boulder position relative to the bus. Thus, the responses appear faster for all the uncertainty cases, and as shown in Table 5 the fuel required to complete the maneuvers and to stabilize the system are minimal (see Table 5).
Figure 8: Nichols Chart for the closed loop FEM model with linear controller.

Figure 9: Time response of bus orientation for scenario 1

Figure 10: Control effort for scenario 1

Table 5: Fuel required to stabilize the system for both scenarios

<table>
<thead>
<tr>
<th>Scenario</th>
<th>4 m case</th>
<th>2 m case</th>
<th>0 m case</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fuel Consumption [kg]</td>
<td>1.6538</td>
<td>1.3817</td>
<td>1.7197</td>
</tr>
<tr>
<td>Scenario 2</td>
<td>4 m case</td>
<td>2 m case</td>
<td>0 m case</td>
</tr>
<tr>
<td>Fuel Consumption [kg]</td>
<td>0.373</td>
<td>0.236</td>
<td>0.09</td>
</tr>
</tbody>
</table>
3. Linear Controller Simulations with Nonlinear Plant

Simulating the linear controller with the full nonlinear dynamic model is essential in understanding the behavior of the controller and evaluating its performance. Figure 13 shows the system response of the nominal boulder case, SDFAST model under the linear controller.

Figure 13: Time response for bus orientation and the control effort for both scenarios using the linear controller with nonlinear plant

Qualitatively, the responses coincide with those for the linear model in Figs. 9, 10, 11, and 12. However, a slight increase in the necessary fuel amount is seen in this case, as shown in Table 6.

B. Nonlinear Control Laws

The attitude controller used here is the robust nonlinear tracking control law proposed in our previous work.\textsuperscript{18} This controller was developed for the attitude dynamics of a rigid body, in the following form.

\[
\dot{\omega} = (\mathbf{I} \omega) \times \omega + Bu + d
\]  

(44)

where, \( \mathbf{I} \) is the inertia tensor of the system, \( \omega \) is the angular velocity, \( u \) is the input torque, and \( d \) is an unknown disturbance torque. Moreover, it is assumed that there is an uncertainty between the known and actual values of \( \mathbf{I} \), measured and actual values of \( \omega \), and commanded and actual values of \( u \).
The tracking problem is to track a desired trajectory \( q_d(t), \dot{q}_d(t) \in \mathbb{R}^3 \) for the bus orientation. The control law can be defined using constant positive definite matrices \( K_c, \Lambda_c \in \mathbb{R}^{3 \times 3} \).

\[
\begin{align*}
    u_c &= \mathbf{I} \dot{\omega}_r - S(\mathbf{I} \omega) \omega_r - K_c (\omega - \omega_r) \\
    \omega_r &= Z^{-1}(q) \dot{q}_d(t) + Z^{-1}(q) \Lambda_c (q_d(t) - q)
\end{align*}
\]  

(45) (46)

Here, \( S(.) \) denotes the skew symmetric cross product matrix of a vector. And \( Z(q) \) is a transform that relates euler angle rates to angular velocity. For our system, \( Z^{-1}(q) = J_{wb} \), where \( J_{wb} \) is the angular velocity Jacobian for the Bus as defined in Eq. (8).

Note that this controller does not use feed-forward cancellation of nonlinear terms in the dynamics, as is the conventional approach. The controller does not completely cancel out the \( S(\mathbf{I} \omega) \omega \) term from the dynamics. This is a crucial property that allows for robustness in the system by dealing with uncertainties in the system inertia tensor. In fact it can be shown that for bounded uncertainties in \( \mathbf{I}, \omega, u \) and bounded disturbance torque \( d \), this control law guarantees global exponential convergence of the system trajectory to a bounded error ball around the desired trajectory. It guarantees global exponential convergence of tracking errors with finite-gain \( L_p \) stability in the presence of modeling uncertainties and disturbances, and it reduces the resultant disturbance torque. Furthermore, this nonlinear control law permits the use of any attitude representation, and its integral control formulation eliminates any constant disturbance.

The ARRM system dynamics are more complicated than the rigid body dynamics of Eq. 44. However, the net effect of all the structural modes in the system may be seen as a disturbance for the bus orientation. That is, all of the inertial forces and reaction torques generated by the rigid bodies and joints in the system may be grouped together as a disturbance torque \( d \) for the rigid body dynamics of the spacecraft bus. Thus, the robust nonlinear tracking control law may be used to stabilize the ARM system. This is assuming that the disturbance torque is somehow bounded, which is a valid assumption for small deflections about the equilibrium configuration.

There are numerous ways to design the time dependent desired trajectory \( q_d(t), \dot{q}_d(t) \) for the controller including several types of optimal trajectories. However, the large modeling uncertainties in the ARM system are more effectively dealt with by a derivative plus proportional-derivative (D+PD) control strategy. In the D+PD strategy, pure derivative control is used to slow down a fast spinning system and once the angular velocity falls below a certain threshold, proportional-derivative control is used to track the desired trajectory.

Such a control strategy can be realized using the controller from Equation 45 by careful design of the desired trajectory. For a fast spinning system, with high angular velocities, the desired trajectory is set such that \( \omega_r = 0 \). This leads to a pure derivative controller.

\[
u_c = -K_r \omega
\]  

(47)

Once the angular velocities fall below a defined threshold, the desired trajectory is changed to \( q_d(t) = q_{\text{final}}, \dot{q}_d(t) = 0 \). This results in the original controller with \( \omega_r = Z^{-1}(q) \Lambda_r (q_{\text{final}} - q) \). Being a switching controller, this approach suffers from chattering about the switching point. This is dealt by using a relay-like threshold where the switching occurs at boundaries of a wider region about the threshold point.

1. Nonlinear Controller Design

The nonlinear control law uses two gains that require tuning, \( K_c \) and \( \lambda_c \). In this section, the tuning procedure for these gains is explained. An iterative search is performed to find values of \( K_c \) and \( \lambda_c \) that lead to desired system response. Through this process, intervals of desirable gains, \( K_c \) and \( \lambda_c \) were identified by the use of two performance metrics: generalized gain margin and system bandwidth.

a. Identifying desirable gain intervals through the use of GGM.
The system is first simulated in scenario 1, for different values of the gains $K_c$ and $\lambda_c$ using a very coarse mesh. This helps to quickly identify the boundaries outside which the controller performance is not up to par. Then the mesh is refined and the simulations are performed for again for varying values of the gains. At this point the GGM gain is unity and the time delay is $\frac{3}{64}$ s.

These simulations give a sense of the qualitative effect each gain has on the controller. For instance, as can be seen in Fig. 14, increasing $\lambda_c$ decreases settling time and leads to low frequency oscillations. Moreover, the steady state error surges along with the fuel required. Interestingly, it is observed that $\lambda_c > 0.5$ leads to a massive jump in control effort and fuel required. On the other hand, increasing $K_c$, speeds up the response and the oscillations start disappearing.

From these simulations, a small interval of $K_c$ and $\lambda_c$ values is identified where the response is convergent. For all of these values the GGM gain where the controller goes unstable is found. Only those controllers with GGM $\leq 5$ are deemed robust for our purposes.

For these robust controllers, the simulations are run again using scenario 2, to check if the GGM or GPM change with the different scenario. This wasn’t seen for any of the robust controllers. The interval with these robust controllers is highlighted in Fig. 14 with the dotted line. Figure 15 shows one example for each response type.

b. Identifying desirable gain intervals through system bandwidth.

Using values of the controller gains around the interval previously identified, the system is simulated to study its response in tracking a sinusoidal reference signal. The bandwidth is determined and a new interval of gain values with acceptable bandwidth is defined. Qualitatively though, it is observed that the bandwidth values does not change with variations in the controller gains but the accuracy of
the tracking the signal changes. The tracking is smoother and more accurate for higher values of the controller gains. This can be seen in Fig. 16 which shows results for tracking of sinusoidal signal along the x-axis. Analogous results are obtained for the other two axes. The double solid lines in Fig. 14 represent the boundaries of this interval.

![Figure 16: Examples of the sinusoidal reference tracking](image)

Gain values, $K_c = 70000$ and $\lambda_c = 0.25$, lie inside both of these intervals. These are the values used for the following simulations of the controller.

### 2. Nonlinear Controller Simulations

Fig. 17 shows the simulation responses for the nonlinear controller with the nonlinear plant for the nominal boulder mass. These simulations are run with the SDFAST model. As shown in the Fig. 17, the responses for both scenarios converge. However, the trajectories look more complicated and control effort larger, when compared to the linear controller in Fig. 13.

Upon further analysis, three dominant low frequency oscillations are identified in this response. The first is along the y-axis and the other two are along the x-axis. These are due to the excitation of certain structural modes of the system. Figure 18 shows these investigated oscillations and the corresponding structural modes.

![Figure 18: Investigated oscillations and structural modes](image)
Figure 17: Time response for Bus orientation and control effort for both scenarios using the nonlinear controller with the nonlinear model

The pitch oscillation frequency corresponds to the seventh natural frequency of the system and occurs only in scenario 1, while the roll oscillation frequencies match with the eleventh and twelfth natural frequencies, which represent the solar array flexing modes. The latter arises in both scenarios.

3. Structural Filter for Nonlinear Controller

To attenuate the roll axis oscillations, a structural filter\(^{41,42}\) affecting the eleventh and twelfth natural modes is developed. It is possible to use this filter, because the modes involving only the solar arrays are not affected by the uncertainty in the model.

\[
G_{\text{filter}} = \frac{(s/z)^2 + 2s/z + 1}{(s/p)^2 + 2\zeta p s/p + 1} \tag{48}
\]

\[z = 2\pi(0.0977) \quad p = 2\pi(0.093) \quad \zeta_p = 0.1\]

This notch filter (Eq. (48)) acts on the output signal from the controller. The responses in Figure 19 demonstrate the effectiveness of the notch filter in attenuating the periodic disturbances.

The nonlinear controller coupled with the notch filter requires more fuel than the linear controller to complete the stabilization maneuvers, as presented in Table 7.

Table 7: Fuel required to stabilize the system using the nonlinear controller with structural filter for both scenarios

<table>
<thead>
<tr>
<th>Fuel Consumption [Kg]</th>
<th>Scenario 1</th>
<th>Scenario 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>62.26</td>
<td>63.81</td>
</tr>
</tbody>
</table>
Figure 18: (a) Oscillations present in the nonlinear controller responses for both scenarios (b) Excited structural mode on the pitch angle in scenario 1 (Mode 7). (c) Excited structural modes on the roll angle in scenarios 1 and 2 (Mode 11-12)

C. Comparison of Control Laws

In this section, both the nonlinear and linear controllers are compared to each other using GGM, GPM and bandwidth.

1. Performance Comparison

Note that the bandwidth estimated through Bode plots is for a SISO subsystem that neglects dynamic coupling in the system. Consequently, the bandwidths calculated through simulations are found to be much smaller than those found through Bode plots. Additionally, the controller behavior is simulated for a simplified MIMO system with only one input, the angle vector. Figure 20 displays the sinusoidal outputs. The bandwidths of this simplified system match the values calculated from the Bode plots, as shown in the Table 8 and Fig. 21.

Table 8: Linear controller bandwidth calculated through simulations with the linear plant

<table>
<thead>
<tr>
<th>Bandwidth [Hz]</th>
<th>MIMO SYSTEM</th>
<th>MIMO SYSTEM</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>3 Inputs 18 Outputs</td>
<td>6 Inputs 18 Outputs</td>
</tr>
<tr>
<td>xx</td>
<td>yy</td>
<td>zz</td>
</tr>
<tr>
<td>4 m case</td>
<td>0.023</td>
<td>0.021</td>
</tr>
<tr>
<td>2 m case</td>
<td>0.041</td>
<td>0.045</td>
</tr>
<tr>
<td>0 m case</td>
<td>0.08</td>
<td>0.156</td>
</tr>
</tbody>
</table>
Figure 19: Time response for Bus orientation and control effort for both scenarios using the nonlinear controller alongwith the structural filter with the nonlinear model

Figure 20: (a) Simplified block diagrams for three input system. (b) Simplified block diagrams for six input system

The same calculations were also carried out for the linear controller with nonlinear plant, as shown in Fig. 22. A similar discrepancy is observed between the results of the two simulation types (see Table 9).

The bandwidth for the nonlinear controller exhibits similar reductions compared with the closed loop Bode plot bandwidth as shown in Fig. 23 and Table 9. Moreover, it is not possible to simulate a simplified subsystem, due to the nonlinear nature of the controller, which requires desired angular rate and acceleration trajectories (Eq. (45)). Therefore, bandwidth is not used as a comparison parameter, eventhough a slight increase in bandwidth is observed for the nonlinear controller over the linear one. Nevertheless, bandwidth is still used to determine the relative performance of controller for different gain values, $K_c$ and $\lambda_c$. 


Figure 21: (a) Sinusoidal tracking responses of the linear controller and the linearized plant with three inputs (angular vector). (b) Sinusoidal tracking responses of the linear controller and the linearized plant with six inputs (angular and angular rates vector)

Table 9: Linear and nonlinear controller bandwidths calculated through simulations with the nonlinear plant

<table>
<thead>
<tr>
<th>Linear Controller</th>
<th>3 inputs 18 outputs</th>
<th>6 inputs 18 outputs</th>
<th>Non-linear Controller</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.023</td>
<td>0.021</td>
<td>0.024</td>
</tr>
<tr>
<td></td>
<td>0.0018</td>
<td>0.0017</td>
<td>0.004</td>
</tr>
<tr>
<td>Non-linear Controller</td>
<td>0.004</td>
<td>0.004</td>
<td>0.004</td>
</tr>
</tbody>
</table>

For nominal boulder case, the linear controller offers $GGM_{\text{min}} = 27$ and $GPM_{\text{min}} = 0.27$. Whereas, the nonlinear controller with structural filter offers a $GGM_{\text{min}} = 5$ and $GPM_{\text{min}} = 0.21$.

To evaluate the robustness of each controller, the non-nominal boulder cases (2 m and 5 m diameter) are used. In Table 10, all the margins for the nominal and non-nominal models are shown. It is interesting to note that the margins for the linear controller remain high even when the system is far away form the linearization point. Figures 24, 25, 26 and 27 show the responses of the controllers for both scenarios.

To further evaluate the performance, both controllers are simulated for challenging scenarios, such as large angle, multi-axis slew maneuvers. In several of the simulated cases, the linear controller fails to converge at large slew angles. On the other hand, the nonlinear controller continues to offer good responses. This can be seen in Figs. 28 and 29.

This failure of the linear controller is a scenario worth further investigations. The failure could be in part due to the singularity in the Euler angle representation. However, the fact that the slew angles are far away from the singularity and that the nonlinear controller demonstrates satisfactory convergence, makes it more likely that the failure experienced is due to the linear controller. Thus, the comparison demonstrates the superiority of the nonlinear controller over the linear controller in handling challenging scenarios.
This paper analyzes the attitude control problem for the ARRM spacecraft after it has captured the boulder using its robotic arms. First, the detailed nonlinear dynamic model of the ARRM spacecraft with the captured boulder is presented, along with frequency domain analysis of the linearized model. Second, linear and nonlinear control laws are designed to stabilize the attitude of the spacecraft-boulder system and track desired attitude trajectories. The proportional-derivative linear controller with lead-lag compensation and roll-off filtering is designed using standard design techniques. The robust nonlinear tracking control law, derived in Ref. 18, is introduced. It guarantees global exponential convergence of tracking errors with finite-gain $L_p$ stability in the presence of modeling uncertainties and disturbances, and it reduces the resultant disturbance torque. The robustness properties of the control laws are demonstrated through simulations of the closed loop system and computation of generalized gain and phase margins. Both control laws are capable of stabilizing the system despite uncertainties in the physical parameters of the boulder for standard scenarios, but only the nonlinear controller is suitable for more challenging scenarios involving large slew.

IV. Conclusion
Table 10: Generalized Gain Margins and Generalized Phase Margins for linear and nonlinear controllers with nominal and non-nominal models

<table>
<thead>
<tr>
<th>Boulder Diameter</th>
<th>Linear Controller</th>
<th>Non-Linear Controller</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 m Scenario 1/2</td>
<td>36 0.2</td>
<td>6 0.14</td>
</tr>
<tr>
<td>4 m Scenario 1/2</td>
<td>27 0.27</td>
<td>5 0.21</td>
</tr>
<tr>
<td>5 m Scenario 1/2</td>
<td>31 0.28</td>
<td>5 0.2</td>
</tr>
</tbody>
</table>

Figure 24: Responses of (a) linear and (b) nonlinear controller with 2 m diameter plant in scenario 1

Figure 25: Responses of (a) linear and (b) nonlinear controller with 5 m diameter plant in scenario 1
Figure 26: Responses of (a) linear and (b) nonlinear controller with 2 m diameter plant in scenario 2

Figure 27: Responses of (a) linear and (b) nonlinear controller with 5 m diameter plant in scenario 2

Figure 28: Responses of (a) linear and (b) nonlinear controller with 5 m diameter plant for a large angle slew maneuver along two axes
angles maneuvers along more than one axis. Finally, we conclude that the robust nonlinear tracking control law is suitable for the attitude control of the ARRM spacecraft with the captured boulder.

Appendix

Positions of center of mass of the manipulator links in the bus frame.

\[
\begin{align*}
  r_{SE1} &= T_{SE1}^B \left[ \frac{-l_{SE}}{2}, 0, 0, 1 \right]^\top \\
  r_{EW1} &= T_{EW1}^B \left[ \frac{-l_{EW}}{2}, 0, 0, 1 \right]^\top \\
  r_{WT1} &= T_{WT1}^B \left[ \frac{-l_{WT}}{2}, 0, 0, 1 \right]^\top \\
  r_{SE2} &= T_{SE2}^B \left[ \frac{l_{SE}}{2}, 0, 0, 1 \right]^\top \\
  r_{EW2} &= T_{EW2}^B \left[ \frac{l_{EW}}{2}, 0, 0, 1 \right]^\top \\
  r_{WT2} &= T_{WT2}^B \left[ \frac{l_{WT}}{2}, 0, 0, 1 \right]^\top
\end{align*}
\]
Linear velocities of the rigid bodies, expressed in the bus frame.

\[
v_B = [R^B_{I, Z}, 0, 0, 0, 0] \dot{q} = J^I_B \dot{\theta} \\
v_P = [R^B_{I, Z}, -S(r_P)J_{\omega_B}, J_{\omega_P}, 0, 0, 0] \dot{q} = J^I_P \dot{\theta} \\
v_N = [R^B_{I, Z}, -S(r_N)J_{\omega_B}, 0, J_{\omega_N}, 0, 0] \dot{q} = J^I_N \dot{\theta} \\
v_{SE1} = [R^B_{I, Z}, -S(r_{SE1})J_{\omega_B}, 0, 0, J_{v_{SE1}}, 0] \dot{q} = J^I_{v_{SE1}} \dot{\theta} \\
v_{EW1} = [R^B_{I, Z}, -S(r_{EW1})J_{\omega_B}, 0, 0, J_{v_{EW1}}, 0] \dot{q} = J^I_{v_{EW1}} \dot{\theta} \\
v_{WT1} = [R^B_{I, Z}, -S(r_{WT1})J_{\omega_B}, 0, 0, J_{v_{WT1}}, 0] \dot{q} = J^I_{v_{WT1}} \dot{\theta} \\
v_{SE2} = [R^B_{I, Z}, -S(r_{SE2})J_{\omega_B}, 0, 0, J_{v_{SE2}}, 0] \dot{q} = J^I_{v_{SE2}} \dot{\theta} \\
v_{EW2} = [R^B_{I, Z}, -S(r_{EW2})J_{\omega_B}, 0, 0, J_{v_{EW2}}, 0] \dot{q} = J^I_{v_{EW2}} \dot{\theta} \\
v_{WT2} = [R^B_{I, Z}, -S(r_{WT2})J_{\omega_B}, 0, 0, J_{v_{WT2}}, 0] \dot{q} = J^I_{v_{WT2}} \dot{\theta} \\
v_{AQ} = [R^B_{I, Z}, -S(r_{AQ})J_{\omega_B}, 0, 0, J_{v_{AQ}}, 0] \dot{q} = J^I_{v_{AQ}} \dot{\theta} \\
v_{AV} = [R^B_{I, Z}, -S(r_{AV})J_{\omega_B}, 0, 0, J_{v_{AV}}, 0] \dot{q} = J^I_{v_{AV}} \dot{\theta}
\]

Angular velocities of the rigid bodies, expressed in the bus frame.

\[
\omega_B = [0, J_{\omega_B}, 0, 0, 0] \dot{\theta} = J^I_B \dot{\theta} \\
\omega_P = [0, J_{\omega_B}, J_{\omega_P}, 0, 0, 0] \dot{\theta} = J^I_P \dot{\theta} \\
\omega_N = [0, J_{\omega_B}, 0, J_{\omega_N}, 0, 0] \dot{\theta} = J^I_N \dot{\theta} \\
\omega_{AQ} = [0, J_{\omega_B}, 0, 0, J_{\omega_{AQ}}, 0] \dot{\theta} = J^I_{\omega_{AQ}} \dot{\theta} \\
\omega_{AV} = [0, J_{\omega_B}, 0, 0, J_{\omega_{AV}}, 0] \dot{\theta} = J^I_{\omega_{AV}} \dot{\theta} \\
\omega_{SE1} = [0, J_{\omega_B}, 0, 0, J_{\omega_{SE1}}, 0] \dot{\theta} = J^I_{\omega_{SE1}} \dot{\theta} \\
\omega_{EW1} = [0, J_{\omega_B}, 0, 0, J_{\omega_{EW1}}, 0] \dot{\theta} = J^I_{\omega_{EW1}} \dot{\theta} \\
\omega_{WT1} = [0, J_{\omega_B}, 0, 0, J_{\omega_{WT1}}, 0] \dot{\theta} = J^I_{\omega_{WT1}} \dot{\theta} \\
\omega_{SE2} = [0, J_{\omega_B}, 0, 0, J_{\omega_{SE2}}, 0] \dot{\theta} = J^I_{\omega_{SE2}} \dot{\theta} \\
\omega_{EW2} = [0, J_{\omega_B}, 0, 0, J_{\omega_{EW2}}, 0] \dot{\theta} = J^I_{\omega_{EW2}} \dot{\theta} \\
\omega_{WT2} = [0, J_{\omega_B}, 0, 0, J_{\omega_{WT2}}, 0] \dot{\theta} = J^I_{\omega_{WT2}} \dot{\theta}
\]
Kinetic energies of the rigid bodies.

\[ T_B = \frac{1}{2} \dot{q}^T [J_B^T m_B J_B^I + J_{\omega_B}^T I_B J_{\omega_B}] \dot{q} \]  

\[ T_P = \frac{1}{2} \dot{q}^T [J_P^T m_P J_P^I + (R_B^P J_{\omega_P})^T I_P (R_B^P J_{\omega_P})] \dot{q} \]  

\[ T_N = \frac{1}{2} \dot{q}^T [J_N^T m_N J_N + (R_B^N J_{\omega_N})^T I_N (R_B^N J_{\omega_N})] \dot{q} \]  

\[ T_{AQ} = \frac{1}{2} \dot{q}^T [J_{AQ}^T m_{AQ} J_{AQ}^I + (R_B^{AQ} J_{\omega_{AQ}})^T I_{AQ} (R_B^{AQ} J_{\omega_{AQ}})] \dot{q} \]  

\[ T_{AV} = \frac{1}{2} \dot{q}^T [J_{AV}^T m_{AV} J_{AV}^I + (R_B^{AV} J_{\omega_{AV}})^T I_{AV} (R_B^{AV} J_{\omega_{AV}})] \dot{q} \]  

\[ T_{SE1} = \frac{1}{2} \dot{q}^T [J_{SE1}^T m_{SE1} J_{SE1}^I + (R_B^{SE1} J_{\omega_{SE1}})^T I_{SE1} (R_B^{SE1} J_{\omega_{SE1}})] \dot{q} \]  

\[ T_{EW1} = \frac{1}{2} \dot{q}^T [J_{EW1}^T m_{EW1} J_{EW1}^I + (R_B^{EW1} J_{\omega_{EW1}})^T I_{EW1} (R_B^{EW1} J_{\omega_{EW1}})] \dot{q} \]  

\[ T_{WT1} = \frac{1}{2} \dot{q}^T [J_{WT1}^T m_{WT1} J_{WT1}^I + (R_B^{WT1} J_{\omega_{WT1}})^T I_{WT1} (R_B^{WT1} J_{\omega_{WT1}})] \dot{q} \]  

\[ T_{SE2} = \frac{1}{2} \dot{q}^T [J_{SE2}^T m_{SE2} J_{SE2}^I + (R_B^{SE2} J_{\omega_{SE2}})^T I_{SE2} (R_B^{SE2} J_{\omega_{SE2}})] \dot{q} \]  

\[ T_{EW2} = \frac{1}{2} \dot{q}^T [J_{EW2}^T m_{EW2} J_{EW2}^I + (R_B^{EW2} J_{\omega_{EW2}})^T I_{EW2} (R_B^{EW2} J_{\omega_{EW2}})] \dot{q} \]  

\[ T_{WT2} = \frac{1}{2} \dot{q}^T [J_{WT2}^T m_{WT2} J_{WT2}^I + (R_B^{WT2} J_{\omega_{WT2}})^T I_{WT2} (R_B^{WT2} J_{\omega_{WT2}})] \dot{q} \]  

References


