Nonlinear Attitude Control of Spacecraft with a Large Captured Object

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This paper presents an attitude control strategy and a new nonlinear tracking controller for a spacecraft carrying a large object, such as an asteroid or a boulder. If the captured object is larger or comparable in size to the spacecraft and has significant modeling uncertainties, conventional nonlinear control laws that use exact feedforward cancellation are not suitable because they exhibit a large resultant disturbance torque. The proposed nonlinear tracking control law guarantees global exponential convergence of tracking errors with finite-gain $L_2$ stability in the presence of modeling uncertainties and disturbances, and it reduces the resultant disturbance torque. Furthermore, this control law permits the use of any attitude representation, and its integral control formulation eliminates any constant disturbance. Under small uncertainties, the best strategy for stabilizing the combined system is to track a fuel-optimal reference trajectory using this nonlinear control law because it consumes the least amount of fuel. In the presence of large uncertainties, the most effective strategy is to track the derivative plus proportional–derivative-based reference trajectory because it reduces the resultant disturbance torque. The effectiveness of the proposed attitude control methods is demonstrated by using results of numerical simulation based on an Asteroid Redirect Mission concept.

Nomenclature

\[
\begin{aligned}
B, \dot{B} &= \text{control influence matrices } (B \in \mathbb{R}^{3\times n_t}), \text{ where } n_t \text{ is number of actuators} \\
B_{CM} &= \text{center of mass of the combined system} \\
d_{ext}, d_{ext,1,2} &= \text{external disturbance torque} \\
\dot{d}_{ext,2} &= \text{resultant disturbance torques} \\
I &= \text{identity matrix} \\
J_{obj}^{B_{CM}}, J_{obj}^{B_{CM}} &= \text{inertia tensors of the object at } B_{CM} \text{ in } \mathcal{F}_B \\
J_{sc}^{B_{CM}}, J_{sc}^{B_{CM}} &= \text{inertia tensors of the spacecraft at } S_{CM} \text{ in } \mathcal{F}_S \\
J_{int}^{B_{CM}}, J_{int}^{B_{CM}} &= \text{inertia tensors of the combined system at } B_{CM} \text{ in } \mathcal{F}_B \\
K, \Lambda, \Gamma, k &= \text{control law gains} \\
m_{obj}, m_{sc} &= \text{mass of object, mass of spacecraft} \\
q, \dot{q}, m_{ac} &= \text{modified Rodrigues parameters } (q \in \mathbb{R}^3) \\
q_d, R_d &= \text{desired attitude trajectory} \\
q_e, \dot{q}_e &= \text{attitude error vector} \\
R, \dot{R} &= \text{rotation matrices } (R \in \mathbb{R}^{3\times3}) \\
R^{B/A} &= \text{vector from } A \text{ to } B \\
S_{CM} &= \text{center of mass of the spacecraft} \\
S_O &= \text{point of contact between the spacecraft and the object} \\
S^3 &= \text{three-sphere } ((x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : \| (x_1, x_2, x_3, x_4) \| = 1) \\
SO(3) &= \text{special orthogonal group } ((R \in \mathbb{R}^{3\times3}) : R^T R = I, \det(R) = 1) \\
u &= \text{actuator output } (u \in \mathbb{R}^{n_t}), \text{ where } n_t \text{ is number of actuators} \\
\Delta(\cdot) &= \text{modeling, measurement or actuator error} \\
\bar{\beta} &= \text{quaternions } (\beta \in \mathbb{R}^4) \\
f_{trans}, f_{ss} &= \text{transient and steady-state errors} \\
\lambda(\cdot) &= \text{eigenvalue of matrix} \\
\omega, \dot{\omega} &= \text{angular velocities } (\omega \in \mathbb{R}^3) \\
\omega_c, \epsilon_{\omega} &= \text{angular velocity error vector} \\
\| \cdot \|_p &= \text{p-vector norm or matrix norm}
\end{aligned}
\]

I. Introduction

Multiple space agencies have announced plans for future small-body exploration and hazard mitigation missions [1–2]. As shown in Fig. 1 [4–5], NASA’s proposed Asteroid Redirect Mission (ARM) aims to capture a near Earth orbit (NEO) asteroid or to pick up a boulder from some bigger asteroid and transport the captured object to the Earth–moon system [6]. This paper is motivated by one of the main control challenges in the proposed ARM concept: despinning and three-axis attitude control of the combined spacecraft and asteroid system. The captured object could be significantly larger and heavier (10 to 100 times by mass) than the spacecraft and could have large uncertainties in its physical model. This control problem also arises in other space applications. For example, a spacecraft tasked with removal of orbital debris has to stabilize the spacecraft–debris combination after capturing the debris [7]. Similarly, small satellites could be launched for the purpose of reviving obsolete satellites already in space or mining them for usable parts [8]. The main control problem in all these applications is that the spacecraft has to stabilize the attitude of the combined system after the spacecraft has captured a target object (e.g., asteroid, debris, and satellite) with large model uncertainties. Moreover, the captured object could be larger than or comparable in size to the spacecraft. In this paper, we present a novel control strategy for addressing this control problem.
Attitude control of a spacecraft with large uncertainty is a topic of intense research. Nonlinear adaptive attitude control strategies are discussed in [9–12]. In [13–15], sliding mode control and robust \( N_{\infty} \) linear control are used for attitude control of spacecraft with uncertainties and disturbances. We show that common nonlinear attitude control tracking laws that use exact feedforward cancellation, similar to feedforward linearization, exhibit a large resultant disturbance torque due to unprecedentedly large modeling uncertainties of the captured object. In contrast, attitude control laws that do not have a feedforward cancellation term (e.g., see [16–19]) experience a much smaller resultant disturbance torque. For the purpose of achieving superior robustness and tracking performance, nonlinear attitude tracking control should be used in lieu of linear control. Therefore, the proposed robust nonlinear tracking control law is designed to exploit the benefit of no feedforward cancellation while achieving superior tracking performance in the presence of large modeling uncertainties, measurement errors, and actuator saturations.

The organization and main contributions of this paper are as follows. Section II discusses the problem statement and some preliminaries. The first contribution of this paper, discussed in Sec. II.A.2, is to compare the resultant disturbance torques for different types of control laws. We show that control laws that use feedforward cancellation experience a large resultant disturbance torque that can lead to actuator saturations. Therefore, such control laws are not suitable for the present control problem.

The second contribution of this paper, discussed in Sec. III, is the development of a new robust nonlinear tracking control law that guarantees global exponential convergence of the system’s attitude trajectory to the desired attitude trajectory and permits the use of any attitude representation. In the presence of bounded disturbances, this control law is finite-gain \( L_p \) stable and input-to-state stable. We show that this control law is related to the well-known tracking control law for Euler–Lagrangian systems [18,20], but the new attitude tracking control law directly prescribes the control torque input with less dependence on the kinematic relationship. Another advantage of this new control law is that it can be easily extended with an integral control term to eliminate constant disturbances while retaining the original global exponential convergence property. Moreover, this exponentially stabilizing attitude control law can be extended to employ \( SO(3) \) for global attitude representation.

In Sec. IV, we discuss techniques for generating fuel-optimal and resultant disturbance torque minimizing desired attitude trajectories. We also outline a framework for reducing the resultant disturbance torque for the new attitude tracking control law.

In Sec. V, we demonstrate our control strategy using results of numerical simulation based on an ARM type. We present a comparative study of the fuel usage and time of convergence of multiple attitude control laws. The third contribution of this paper indicates that the best control strategy under very small modeling uncertainties, which can be achieved using online system identification from both proximity and contact operations, is to track the fuel-optimal reference trajectory using the globally exponentially stable robust nonlinear tracking control law. On the other hand, in the presence of large modeling uncertainties, measurement errors, and actuator saturations, the best control strategy is to have the robust nonlinear tracking control law track a derivative plus proportional–derivative-based desired attitude trajectory. We also present a detailed sensitivity analysis of the robust nonlinear tracking control law to show that the fuel consumed by the conceptual ARM spacecraft using this control strategy is upper bounded by 300 kg for the nominal range of NEO asteroid parameters. This paper is concluded in Sec. VI.

II. Preliminaries and Problem Statement

In this section, we first present the attitude kinematics and dynamics of the combined system in Sec. II.A and then state the main control problem in Sec. II.B.

A. Attitude Dynamics and Kinematics of the Combined System

In this section, we present the attitude dynamics and kinematics equations that are used in this paper. We assume that the combined spacecraft and captured object form a rigid body.

As shown in Fig. 2, the center of mass of the combined system (\( B_{CM} \)) is the origin of the body-fixed frame \( F_B \). Let \( S_D \), which is the point of contact between the spacecraft and the object, be the origin of the spacecraft frame \( F_S \). We assume that attitude orientation of \( F_B \) with respect to \( F_I \) is the same as that of \( F_S \) with respect to \( F_I \) (i.e., the rotation matrix from \( F_S \) to \( F_B \) is an identity matrix).

1. Attitude Dynamics with Uncertainty

Let \( J_{CM}^{BC} \) be the unknown, constant, positive–definite inertia tensor of the captured object at \( B_{CM} \) and expressed in \( F_B \). Let \( J_{CM}^{SC} \) be the known, constant, positive–definite inertia tensor of the spacecraft at the center of mass of the spacecraft (\( S_{CM} \)) and expressed in \( F_S \). Let \( r_{SCM/B_{CM}} \) denote the unknown vector from \( B_{CM} \) to \( S_D \). The combined inertia tensor of the system at \( B_{CM} \), expressed in \( F_B \), is determined using the parallel axis theorem:

\[
J_{CM}^{BC} = J_{CM}^{BC} + J_{CM}^{SC} + m_{SC} (r_{SCM/B_{CM}})^T (r_{SCM/B_{CM}}) I
- (r_{SCM/B_{CM}}) (r_{SCM/B_{CM}})^T
\]

(1)

where \( r_{SCM/B_{CM}} = r_{SCM/S_D} + r_{S_D/B_{CM}} \), \( r_{S_D/S_O} \) is the known vector from \( S_O \) to \( S_{CM} \), \( m_{SC} \) is the mass of the spacecraft; and the rotation matrix from the spacecraft frame to the body frame is an identity matrix.

![Fig. 2 The conceptual ARM spacecraft [6] is shown with a captured asteroid. The frames \( F_I, F_B, \) and \( F_S \) and the center of mass \( B_{CM} \) are shown.](image-url)
Let $\omega \in \mathbb{R}^3$ be the angular velocity of the system in the body-fixed frame $\mathcal{F}_B$ with respect to the inertial frame $\mathcal{F}_I$ and expressed in the frame $\mathcal{F}_B$. Let $u \in \mathbb{R}^n$ be the outputs of $n$ actuators and $B \in \mathbb{R}^{3\times n}$ be the corresponding control influence matrix. The attitude dynamics of the rigid combination is given by

$$J_{\text{tot}}^B \ddot{\omega} = (J_{\text{tot}}^B \omega) \times \omega + Bu + d_{\text{ext}}$$

where $d_{\text{ext}}$ represents the external torque acting on the system.

We now study the effect of modeling uncertainties in $J_{\text{tot}}^B$ and $r_{SO}^B$, measurement errors in $\omega$, and actuator errors in $u$ on the attitude dynamics of the system [Eq. (2)]. Let $J_{\text{tot}}^B = J_{\text{obj}}^B + \Delta J_{\text{CM}}^B$, where $(\cdot)$ and $\Delta (\cdot)$ refer to the known and uncertain parts, respectively. Similarly, let $r_{SO}^B = r_{SO}^B + \Delta r_{SO}^B$, $\omega = \omega + \Delta \omega$, and $u = \hat{u} + \Delta u$. Because of these uncertainties, the combined inertia tensor is given by

$$J_{\text{tot}}^B = J_{\text{tot}}^B + \Delta J_{\text{tot}}^B$$

Similarly, the control influence matrix can be decomposed into $B = \hat{B} + \Delta B$ because it depends on $r_{SO}^B$. Simplifying the dynamics of the system [Eq. (2)] gives

$$J_{\text{tot}}^B (\dot{\omega} + \Delta \dot{J}_{\text{tot}}^B) = J_{\text{tot}}^B \omega + \Delta \dot{J}_{\text{tot}}^B \omega \times \dot{\omega} = u_c + d_{\text{res}}$$

Example 1: In Table 1, we now numerically compare the magnitude of some of the terms in the resultant disturbance torques $d_{\text{res}1}$ in Eq. (4), $d_{\text{res}2}$ in Eq. (6), and $d_{\text{res}3}$ in Eq. (7) using the following values:

$$\begin{align*}
J_{\text{tot}}^B &= 10^6 \times \begin{bmatrix}
1.2652 & 0.4397 & 0.0015 \\
0.4397 & 3.8688 & 0.0002 \\
0.0015 & 0.0002 & 3.5440
\end{bmatrix} \text{kg} \cdot \text{m}^2, \\
\omega &= [0.01, 0.02, 0.03] \text{rad/s}, \\
\dot{\omega} &= [0.010, 0.020, 0.0299] \text{rad/s}, \\
\Delta \omega &= 10^{-4} \times [-0.44, 0.09, 0.91] \text{rad/s}
\end{align*}$$

and we have neglected $\dot{\omega}$ and $\Delta \omega$ because they are much smaller than $\omega$.

The magnitude of the disturbance term $(\Delta J_{\text{tot}}^B \dot{\omega} \times \omega)$ in Table 1 is significantly larger than the magnitude of other disturbance terms because of unprecedentedly large modeling uncertainties in the captured object. Moreover, this resultant disturbance torque is so large that it is comparable to the maximum control torque that the spacecraft can generate. Hence, control laws that have the disturbance term $(\Delta J_{\text{tot}}^B \dot{\omega} \times \omega)$ in their resultant disturbance torque, like the feedforward-cancellation-based control law Eq. (6), are not suitable for this control problem.

2. Impact of Feedforward Cancellation on Resultant Disturbance Torque

In this section, we compare the resultant disturbance torques of different attitude control laws. If a linear control law (e.g., proportional–derivative control) is used, then the resultant disturbance torque is given by $d_{\text{res}}$ in Eq. (4). However, a linear control law does not achieve global exponential stability for attitude tracking. If the following feedback linearization-based control law is used:

$$u_c = \hat{J}_{\text{obj}}^B \dot{\omega} - \hat{J}_{\text{CM}}^B \dot{\omega} \times \omega - K_f (\dot{\omega} - \omega_c)$$

where $K_f \in \mathbb{R}^{3 \times 3}$ is a positive–definite constant matrix and $\omega_c$ is the desired reference trajectory, then this control law globally exponentially stabilizes the left-hand side of the following closed-loop system:

$$\dot{\hat{J}}_{\text{CM}}^B (\omega - \omega_c) + K_f (\dot{\omega} - \omega_c) = [d_{\text{res},1} \Delta \hat{J}_{\text{CM}}^B \dot{\omega} - \Delta \hat{J}_{\text{CM}}^B \dot{\omega} \times \omega]$$

and the term $d_{\text{res},1}$ appears as a resultant disturbance torque. Suppose there exists a control law such that it globally exponentially stabilizes the left-hand side of the following closed-loop system:

$$\hat{J}_{\text{CM}}^B (\dot{\omega} - \omega_c) - (\hat{J}_{\text{CM}}^B \omega) \times (\dot{\omega} - \omega_c) + K_f (\dot{\omega} - \omega_c) = [d_{\text{res},2} \Delta \hat{J}_{\text{CM}}^B \dot{\omega} - \Delta \hat{J}_{\text{CM}}^B \dot{\omega} \times \omega]$$

then the term $d_{\text{res},2}$ appears as a resultant disturbance torque. We show later that Eq. (7) is the closed-loop system of our proposed nonlinear tracking control law.

### Table 1: Magnitude of some of the disturbance terms in the resultant disturbance torques

<table>
<thead>
<tr>
<th>Disturbance term</th>
<th>Magnitude (((\cdot)^2)-norm)</th>
<th>Disturbance term present in</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta J_{\text{tot}}^B \dot{\omega} \times \omega$</td>
<td>$372.8 \text{ N} \cdot \text{m}$</td>
<td>X</td>
</tr>
<tr>
<td>$\Delta J_{\text{tot}}^B \dot{\omega} \times \omega_c$</td>
<td>Depends on $\omega_c$</td>
<td>X</td>
</tr>
<tr>
<td>$J_{\text{tot}}^B \Delta \dot{\omega} \times \omega$</td>
<td>$7.8 \text{ N} \cdot \text{m}$</td>
<td>✓</td>
</tr>
<tr>
<td>$J_{\text{tot}}^B \Delta \dot{\omega} \times \Delta \omega$</td>
<td>$6.3 \text{ N} \cdot \text{m}$</td>
<td>✓</td>
</tr>
<tr>
<td>$\Delta J_{\text{CM}}^B \dot{\omega} \times \Delta \omega$</td>
<td>$10.1 \text{ N} \cdot \text{m}$</td>
<td>✓</td>
</tr>
<tr>
<td>$\Delta J_{\text{CM}}^B \dot{\omega} \times \omega$</td>
<td>$7.3 \text{ N} \cdot \text{m}$</td>
<td>✓</td>
</tr>
</tbody>
</table>
Table 2 Properties of attitude representations (adapted from [21])

<table>
<thead>
<tr>
<th>Attitude representation</th>
<th>Range, transformation</th>
<th>Global?</th>
<th>Unique?</th>
</tr>
</thead>
<tbody>
<tr>
<td>Euler angles $\phi, \theta, \psi \in [-\pi, \pi]$</td>
<td>No (singularity at $\theta = \pm \pi/2$)</td>
<td>No</td>
<td></td>
</tr>
<tr>
<td>Euler axis of rotation and angle $e \in S^2, \Phi \in [-\pi, \pi]$</td>
<td>Yes</td>
<td>No ($e^t = -e$, $\Phi^t = 2\pi - \Phi$)</td>
<td>Yes</td>
</tr>
<tr>
<td>Quaternions $\beta \in S^3$</td>
<td>No (singularity at $\Phi = \pm \pi$)</td>
<td>Yes</td>
<td></td>
</tr>
<tr>
<td>$\beta_i = e_i \sin \frac{\Phi}{2}, i \in {1, 2, 3}$</td>
<td></td>
<td>No ($\beta^t = -\beta$)</td>
<td></td>
</tr>
<tr>
<td>Classical Rodrigues parameters $\sigma \in \mathbb{R}^3$</td>
<td>No (singularity at $\Phi = \pm \pi$)</td>
<td>Yes (when $\Phi \neq \pm \pi$)</td>
<td>Yes</td>
</tr>
<tr>
<td>Modified Rodrigues parameters $q \in \mathbb{R}^3$</td>
<td>No (singularity at $\Phi = \pm 2\pi$)</td>
<td>No, $q^t = -e \tan((2\pi - \Phi)/4)$</td>
<td></td>
</tr>
<tr>
<td>Rotation matrix $R \in SO(3)$, det($R$) = 1</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\(\dot{q} = Z(q)\omega\), where 

\[Z(q) = \frac{1}{2} \left[ I - \frac{q q^T}{\|q\|^2} \right] + q q^T + S(q)\] 

\[S(q) = \begin{bmatrix} 0 & -q_3 & q_2 \\ q_3 & 0 & -q_1 \\ -q_2 & q_1 & 0 \end{bmatrix}\] 

(9)

\[\hat{R} = RS(\omega)\] 

(10)

The attitude kinematics equations using Euler angles ($\phi, \theta, \psi$), classical Rodrigues parameters ($\sigma$), and the first three elements of a quaternion vector ($\beta_i$) can also be written in the form of \(\dot{q} = Z(q)\omega\) [like Eq. (9)] with a different definition of $Z(q)$ [25]. We show later that our proposed nonlinear control law permits the use of any attitude representation.

### B. Problem Statement: Attitude Control of the Combined System

The salient features of the attitude control problem discussed in this paper are as follows.

1) The rigid combined system, comprising the spacecraft and the captured object, is tumbling. The tumbling rate can be nonuniform due to the presence of inertia tensor.

2) The object’s inertia tensor, mass, center of mass, and center of gravity have large uncertainties (approximately 10% of the nominal value).

3) The object is noncollaborative (i.e., no actuators are placed on the object). All actuators are onboard the spacecraft.

Let $q_{\text{final}}$ denote the desired attitude orientation of the stabilized system. The attitude control objective is to stabilize the system, in the presence of uncertain physical parameters, bounded disturbances, measurement errors, and actuator saturations, such that for some appropriate $\epsilon_{\text{trans}} > 0$, $\epsilon_{\text{ss}} > 0$, and $T \gg 0$,

\[\|\omega(t)\|_2 \leq \epsilon_{\text{trans}}, \quad \forall \ t > 0\] 

(11)

\[\|q(t) - q_{\text{final}}\|_2 \leq \epsilon_{\text{ss}}, \quad \forall \ t > T\] 

(12)

The transient error bound $\epsilon_{\text{trans}}$ is imposed on the angular velocity $\omega(t)$ in Eq. (11) to ensure that the system is always within the technological capability of the sensors and actuators onboard the spacecraft. It is desired that after time $T$, the system should achieve the desired attitude orientation $q_{\text{final}}$ as shown in the steady-state condition [Eq. (12)]. Note that if the system has to hold its attitude within the given steady-state error bound $\epsilon_{\text{ss}}$, then the desired angular velocity $q_{\text{final}}$ of the stabilized system should be sufficiently close to 0 rad/s.

In this paper, a control law that guarantees global exponential convergence or a contracting closed-loop dynamics in the sense of Lemma 6 is derived to achieve the objectives in Eqs. (11) and (12) by using Lemma 7 (see the Appendix). Hence, in the presence of disturbances, such a globally exponentially stabilizing control law yields finite-gain $L_p$ stability and input-to-state stability [26]. It is a control law that only yields global asymptotic convergence (without any disturbance), then the error in the system’s trajectory may not be bounded for a certain class of disturbance, and proving robustness is more involved [26].

### III. Control Laws for Nonlinear Attitude Control

In this section, we present the new nonlinear attitude tracking control laws that are deemed suitable for satisfying the control problem statement. We first present a novel robust nonlinear tracking control law that guarantees globally exponential convergence of the system’s attitude trajectory to the desired attitude trajectory. To highlight the advantages of this new control law, we also present several extensions of this attitude tracking control law, like augmenting it with an integral control term and deriving an exponentially stabilizing tracking control law on SO(3).

#### A. Robust Nonlinear Tracking Control Law with Global Exponential Stability

The following theorem states the proposed robust nonlinear tracking control law. Note that this control law does not cancel the term $S(J_{\text{axis}}^w \dot{\omega})\omega$, exactly, in contrast with most conventional nonlinear tracking control laws using feedforward cancellation. Although this control law is written for MRP, it can also be used with other attitude representations like Euler angles, classical Rodrigues parameters, and the quaternion vector, by changing the definition of $Z(q)$ [25].

**Theorem 1:** For the given desired attitude trajectory $q_d(t)$, and positive–definite constant matrices $K_r \in \mathbb{R}^{3 \times 3}$ and $\Lambda_r \in \mathbb{R}^{3 \times 3}$, we define the following control law:

\[u_r = J_{\text{axis}}^w(\dot{q}_d)\omega_r - S(J_{\text{axis}}^w \dot{\omega})\omega_r - K_r(\dot{\omega} - \omega_r)\]

where

\[\omega_r = Z^{-1}(\dot{q})\dot{q}_d(t) + Z^{-1}(\dot{q})\Lambda_r(q_d(t) - \dot{q})\] 

(13)

This control law stabilizes the combined system [Eq. (4)] and has the following properties.

1) In the absence of resultant disturbance torque $d_{\text{res,2}}$, this control law guarantees global exponential convergence of the system’s trajectory to the desired trajectory $q_d(t)$.

2) In the presence of bounded resultant disturbance torque $d_{\text{res,2}}$, this control law guarantees that the tracking error $q_e = \dot{q} - q_d$ globally exponentially converges to the following ball:

\[
\lim_{t \to \infty} \int_0^t \|\dot{q}_e\|_2 \leq \frac{\lambda_{\text{max}}(K_r)\lambda_{\text{max}}(J_{\text{axis}}^w)}{\lambda_{\text{min}}(K_r)\lambda_{\text{min}}(J_{\text{axis}}^w)} \times (\sup_{t} \sigma_{\lambda_{\text{max}}}(Z(\dot{q}))) (\sup_{t} \|d_{\text{res,2}}\|_2)
\]

Hence, this control law is finite-gain $L_p$ stable and input-to-state stable (ISS), which are sufficient conditions for satisfying the control problem statements Eqs. (11) and (12).

**Proof:** The closed-loop dynamics, which is obtained by substituting $u_r$ from Eq. (13) into Eq. (4), becomes
\[
J_{tot}^{\text{LCM}} \dot{\omega}_r - S(J_{tot}^{\text{LCM}} \dot{\omega}_r) + K_r \omega_r = \left[ d_{\text{res.2}} + \Delta J_{\text{LCM}}^{\text{LCM}} \dot{\omega}_r - S(\Delta J_{\text{LCM}}^{\text{LCM}} \dot{\omega}_r) \right]_{\text{tot}}
\]

where \( \omega_r = (\dot{\omega} - \omega_r) \). We first show that the control law indeed globally exponentially stabilizes the closed-loop system without the resultant disturbance \( d_{\text{res.2}} \). The virtual dynamics of \( y \), derived from Eq. (14) without \( d_{\text{res.2}} \), is given as

\[
J_{\text{LCM}}^{\text{LCM}} \dot{y} - S(J_{\text{LCM}}^{\text{LCM}} \dot{y}) + K_y y = 0
\]

where \( y = \omega_r \) and \( y = 0 \) as its two particular solutions. After we obtain the dynamics of the infinitesimal displacement at fixed time, \( \delta y \), from Eq. (15), we perform the squared-length analysis (see the Appendix):

\[
\frac{d}{dt} (\delta y^T J_{\text{LCM}}^{\text{LCM}} \delta y) = -2 \delta y^T K_y \delta y \leq -\frac{2 \lambda_{\text{min}}(K_y)}{\lambda_{\text{max}}(J_{\text{LCM}}^{\text{LCM}})} \delta y^T J_{\text{LCM}}^{\text{LCM}} \delta y
\]

where we exploited the skew-symmetric property of the matrix \( S(J_{\text{LCM}}^{\text{LCM}} \dot{\omega}_r) \). Hence, it follows from the contraction analysis (Lemma 6 in the Appendix) that all system trajectories of Eq. (15) converge exponentially fast to a single trajectory (i.e., \( \delta y \to 0 \) and \( \omega_r \to 0 \)) at a rate of \( \lambda_{\text{min}}(K_y) \). Therefore, the family of bounded resultant disturbance \( d_{\text{res.2}} \), it follows from Lemma 7 in the Appendix that

\[
\lim_{t \to \infty} \int_{0}^{t} \| \delta y \|^2_2 \leq \frac{\lambda_{\max}(J_{\text{LCM}}^{\text{LCM}})}{\lambda_{\min}(K_y)} \frac{\sup \| d_{\text{res.2}} \|_2}{2}
\]

where \( \delta q = (\dot{q} - \hat{q}_r) \). In the absence of \( \omega_r \), all system trajectories of \( \delta q \) will converge exponentially fast to a single trajectory (i.e., \( \delta q \to 0 \)) with a rate of \( \lambda_{\min}(\Lambda_r) \), where the virtual displacement \( \delta q \) is an infinitesimal displacement at fixed time. In the presence of \( \omega_r \), it follows from Lemma 7 in the Appendix that

\[
\lim_{t \to \infty} \int_{0}^{t} \| \delta q \|^2_2 \leq \frac{1}{\lambda_{\min}(\Lambda_r)} \frac{\sup \| Z(\hat{q}) \omega_r \|_2}{2}
\]

(19)

Hence, we have shown, by constructing a hierarchically combined closed-loop system of \( \omega_r \) and \( q \), that the attitude trajectory \( q \) will globally exponentially converge to a bounded error ball around the desired trajectory \( \hat{q}_r(t) \). Moreover, it follows from Lemma 7 in the Appendix that this control law is finite-gain \( L_p \) stable and input-to-state stable. Hence, the control gains \( K_r \) and \( \Lambda_r \) can be designed such that the error bounds \( \epsilon_r \) and \( \epsilon_q \) in Eqs. (11) and (12) are satisfied.

The desired attitude trajectory \( \hat{q}_r(t) \) can be any reference trajectory that we would like the system to track. We discuss methods for designing these desired attitude trajectories in Sec. IV.

B. Relation to Nonlinear Tracking Control Using Euler–Lagrangian Systems

In this section, we compare the robust nonlinear tracking control law Eq. (13) with the well-known robust nonlinear tracking control for Euler–Lagrangian (EL) systems [20]. We first state the EL system with uncertainty, which is a combined representation of the attitude kinematics and dynamics of the system:

\[
\hat{M}(\hat{q}) \dot{\hat{q}} + \hat{C}(\hat{q}, \dot{\hat{q}}) \dot{\hat{q}} = \hat{r}_c + \tau_{\text{res.}}
\]

where \( \hat{r}_c = Z^{-1}(\hat{q}) u_c \), \( \hat{M}(\hat{q}) = Z^{-1}(\hat{q}) J_{\text{LCM}}^{\text{LCM}} Z^{-1}(\hat{q}) \),

\[
\dot{\hat{C}}(\hat{q}, \dot{\hat{q}}) = -Z^{-1}(\hat{q}) J_{\text{LCM}}^{\text{LCM}} Z^{-1}(\hat{q}) \dot{Z}(\hat{q}) Z^{-1}(\hat{q})
\]

and \( \tau_{\text{res.}} \) is the resultant disturbance torque acting on the EL system. Note that \( \hat{M}(\hat{q}) - 2\hat{C}(\hat{q}, \dot{\hat{q}}) \) in Eq. (20) is a skew-symmetric matrix, and this property is essential to the stability proof. We use a slight modification of the original robust nonlinear tracking control law Eq. (13), which is given by

\[
u_c = J_{\text{LCM}}^{\text{LCM}} \dot{\omega}_r - S(J_{\text{LCM}}^{\text{LCM}} \dot{\omega}_r) - Z^{-1}(\hat{q}) \dot{K}_r \dot{Z}(\hat{q}) Z^{-1}(\hat{q})
\]

where \( \dot{K}_r = \dot{Z}(\hat{q}) \dot{Z}(\hat{q}) Z^{-1}(\hat{q}) \), and multiplying both sides with \( Z^{-1}(\hat{q}) \) gives

\[
\hat{r}_c = \hat{M}(\hat{q}) \dot{q}_r + \hat{C}(\dot{q}_r, \dot{q}_r) \dot{q}_r - K_r \dot{q}_r - \dot{q}_r
\]

Remark 1 [Advantages of Eq. (13) over the control law for EL systems Eq. (22)]: First, the control law for the EL system [Eq. (22)] extensively uses the measured attitude \( \hat{q} \) and its rate \( \dot{\hat{q}} \) but does not explicitly use the measured angular velocity \( \dot{\hat{q}} \). Moreover, the matrices \( Z(\hat{q}) \) and \( Z^{-1}(\hat{q}) \) might be susceptible to large fluctuations due to measurement errors in \( \hat{q} \), which are used multiple times in Eq. (22). For example, the actual control input \( u_c \) depends on the computed control signal \( \hat{r}_c \), through the relation \( u_c = Z(\hat{q}) \hat{r}_c \). On the other hand, the original control law [Eq. (13)] directly computes \( u_c \).

Second, as shall be seen in Sec. III.C, the stability proof is constructed using a constant matrix \( J_{\text{LCM}}^{\text{LCM}} \) not the nonlinear matrix \( \hat{M}(\hat{q}) \), thereby allowing for an integral control formulation. Third, in Eqs. (21) and (22), the terms \( Z(\hat{q}) K_r Z(\hat{q}) \) and \( \hat{C}(\hat{q}, \dot{\hat{q}}) \) strongly couple the three axes motions using the highly nondiagonal, nonsymmetric matrix \( Z(\hat{q}) \). This strong coupling of the three-axis rotational motions might be undesirable. For example, initially, there might be an error in only one axis, but this coupling will subsequently introduce errors in all three axes. Depending on the inertia matrix, this strong coupling of three-axis motions can be avoided in the proposed control law Eq. (13).

C. Robust Nonlinear Tracking Control Law with Integral Control

Another benefit of the original robust nonlinear tracking control law Eq. (13) is that it can be augmented with an integral control term in a straightforward manner to eliminate any constant external disturbance while ensuring exponential convergence of the system’s attitude trajectory to the desired attitude trajectory.

Theorem 2: For the given desired attitude trajectory \( q_d(t) \), positive–definite constant matrices \( K_m \in \mathbb{R}^{3 \times 3} \) and \( \Lambda_m \in \mathbb{R}^{3 \times 3} \), and (possibly time-varying) uniformly positive–definite diagonal matrix \( K_I(t) \in \mathbb{R}^{3 \times 3} \), we define the following control law:

\[
u_c = J_{\text{LCM}}^{\text{LCM}} \dot{\omega}_r - S(J_{\text{LCM}}^{\text{LCM}} \dot{\omega}_r) \omega_r - K_m (\omega_r - \omega_r) - \int_0^t K_I(\omega_r - \omega_r) \, dt
\]

where \( \omega_r = Z^{-1}(\hat{q}) \hat{q}_r(t) + Z^{-1}(\hat{q}) \Lambda_m (q_d(t) - \hat{q}_r) \).

This control law has the following properties.

1) This control law guarantees global exponential convergence of the system’s trajectory to \( q_d(t) \) for any constant external disturbance (constant bias) acting on the system.
2) In the presence of time-varying disturbance $d_{res,2}$ with a bounded rate $\dot{d}_{res,2}$, this control law guarantees that $q(t)$ will globally exponentially converge to an error ball around $q_d(t)$, whose size is determined by $\dot{d}_{res,2}$ (i.e., finite-gain $L_\infty$ stable and ISS with respect to disturbance inputs within bounded rates).

**Proof:** The closed-loop dynamics is given by

$$J_{tot}^{bCM}\dot{\omega}_e - S(J_{tot}^{bCM})\omega_e + K_m\omega_e + \int_0^t K_i\omega_e \, dt = d_{res,2}$$

(24)

where $\omega_e = (\dot{\omega} - \omega_0)$, and $d_{res,2}$ is defined in Eq. (14). We first show that this control law can eliminate a constant external disturbance; hence, replacing $d_{res,2}$ in Eq. (24) with a constant disturbance term $d_{const}$ gives us

$$J_{tot}^{bCM}\dot{\omega}_e - S(J_{tot}^{bCM})\omega_e + K_m\omega_e + \int_0^t K_i\omega_e \, dt = d_{const}$$

(25)

Differentiating Eq. (25) with respect to time and setting $\dot{d}_{const} = 0$, we get

$$J_{tot}^{bCM}\ddot{\omega}_e + (K_m - S(J_{tot}^{bCM})\omega_e + (K_1 - S(J_{tot}^{bCM}))\omega_e = 0$$

(26)

If we show that Eq. (26) is contracting, then we prove our claim 1, that the given control law can successfully eliminate any constant external disturbance acting on the system. To prove Eq. (26) is globally exponentially stable, we consider two cases that depend on the time-varying nature of the matrix $K_i$.  

Equation (30) is satisfied by $b < b_4$, where $b_4$ is given by

$$b_4 = \frac{\lambda_{\max}(K_1)}{\lambda_{\min}(K_1)I_{sym}}$$

(31)

We first consider the case where $K_1$ is a constant positive–definite diagonal matrix. The matrix $K_1$ can be decomposed into $K_1 = K_1^K K_1^T$, where the matrix $K_1^K$ is also a constant positive–definite diagonal matrix. We introduce the term $y_1$, where $\dot{y}_1$ is defined as $\dot{y}_1 = K_1^T \dot{\omega}_e$. Then, we can write $\dot{\omega}_e$ as

$$\dot{\omega}_e = -(J_{tot}^{bCM})^{-1}(K_m - S(J_{tot}^{bCM}))\omega_e - (J_{tot}^{bCM})^{-1} K_1^T y_1$$

(27)

Note that differentiating Eq. (27) with respect to time and substituting $\dot{y}_1$ gives us Eq. (26). Therefore, these equations can be written in matrix form as

$$\begin{bmatrix} \dot{\omega}_e \\ \dot{y}_1 \end{bmatrix} = \begin{bmatrix} -(J_{tot}^{bCM})^{-1}(K_m - S(J_{tot}^{bCM})) & -(J_{tot}^{bCM})^{-1} K_1^T \\ K_1^T \\ 0 \end{bmatrix} \begin{bmatrix} \omega_e \\ y_1 \end{bmatrix}$$

(28)

We define the positive–definite matrix

$$\Xi = \begin{bmatrix} J_{tot}^{bCM} & bI \\ bI & I \end{bmatrix}$$

where $b$ is a constant between $0 < b < \frac{1}{\lambda_{\max}(K_1)}$. The symmetric matrix $(\Xi F)^{sym} = \frac{1}{2}((\Xi F) + (\Xi F)^T)$ is given by

$$(\Xi F)^{sym} = \frac{1}{2} \begin{bmatrix} K_1 m + K_1 b & -b K_1^T \\ -b K_1 & (J_{tot}^{bCM})^{-1} K_1^T + (J_{tot}^{bCM})^{-1} \end{bmatrix}$$

The sufficient conditions for the matrix $(\Xi F)^{sym}$ to be negative–definite are [27]

$$\frac{K_1}{2} + b K_1^T < 0, \quad -b (J_{tot}^{bCM})^{-1} K_1^T + (J_{tot}^{bCM})^{-1} < 0$$

(29)

$$\lambda_{\max}(-\frac{K_1}{2} + b K_1^T) \lambda_{\min}(-\frac{b}{2} (J_{tot}^{bCM})^{-1} K_1^T + (J_{tot}^{bCM})^{-1}) > 1$$

(30)

Equation (29) is satisfied by

$$0 < b < \frac{\lambda_{\min}(K_1 m + K_1^T)}{2 \lambda_{\max}(K_1^T)}$$

Therefore, the matrix $(\Xi F)^{sym}$ is negative–definite if $b$ is chosen such that

$$0 < b < \min\left(\lambda_{\max}(J_{tot}^{bCM}), \frac{\lambda_{\min}(K_1 m + K_1^T)}{2 \lambda_{\max}(K_1^T)}\right)$$

We define the generalized virtual displacement $\delta z = [\delta \omega_e, \delta y_1]^T$, where $\delta \omega_e$ and $\delta y_1$ are infinitesimal displacements at fixed time. Therefore,

$$\frac{d}{dt}(\delta z^T \Xi \delta z) = \delta z^T ((\Xi F) + (\Xi F)^T) \delta z$$

$$\leq 2 \lambda_{\max}((\Xi F)^{sym}) \|\delta z\|_2^2 \leq \frac{2 \lambda_{\max}((\Xi F)^{sym})}{\lambda_{\max}(\Xi)} (\delta z^T \Xi \delta z)$$

(32)

Hence, it follows from the contraction analysis (Lemma 6 in the Appendix) that all system trajectories converge exponentially fast to a single trajectory ($\delta z \rightarrow 0$ and $\delta \omega_e \rightarrow 0$) at a rate of $-\lambda_{\max}((\Xi F)^{sym})$ and $\|\delta z\|_2$. Moreover, in the presence of bounded time-varying resultant disturbance $d_{res,2}$ with bounded $\dot{d}_{res,2}$, we get from Lemma 7 in the Appendix

$$\lim_{t \rightarrow \infty} \int_0^t \|\delta \omega_e\|_2 \leq \frac{(b + 1) \lambda_{\max}(\Xi)}{-\lambda_{\max}((\Xi F)^{sym})} (sup \lambda_{\max}(\Xi)) (sup \|\dot{d}_{res,2}\|_2)$$

(33)
where \( \| \delta \omega \|_2 \leq \| \delta z \|_2 \) and \( \lambda_{\text{max}}(\Xi) > 1 \) are used. Also, note that the disturbance term in the right-hand side of Eq. (28) is \( 0; -K_1^{-1} \hat{d}_{\text{res,2}} \). The fact that convergence of \( \omega_s \rightarrow 0 \) implies convergence of the system’s trajectory to the desired trajectory \((\hat{q} \rightarrow q_i)\) is already presented in the proof of Theorem 1. This completes the proof.

If both \( K_t \) and \( K_i \) are uniformly positive–definite diagonal matrices, there exits a simpler proof, which is presented here. The matrix \( K_t \) can also be decomposed into \( K_t = K_t^1 K_t^2 \). We introduce another term \( y_2 \), where

\[
y_2 = K_t^1 \omega_s - K_t^{-1} \hat{d}_{\text{res,2}} \tag{34}
\]

Once again, \( \omega_s \) can be written in a form similar to that of Eq. (27). The matrix form of these equations is given by

\[
\begin{bmatrix}
\dot{\omega}_s \\
y_2
\end{bmatrix}
= 
\begin{bmatrix}
-\left(J_{\text{rot}}^{R,CM}\right)^{-1}\left(K_m - S(J_{\text{tot}}^{R,CM}) \omega_s \right) - \left(J_{\text{rot}}^{R,CM}\right)^{-1} K_t^1 \\
K_t^1
\end{bmatrix}
- K_t^{-1} \hat{d}_{\text{res,2}} \tag{35}
\]

Clearly, the symmetric part of the matrix \( \Xi \hat{F} \) is negative–definite. Therefore,

\[
\frac{d}{dt} \left( \delta z^T \Xi \delta z \right) = \delta z^T (\Xi \hat{F} + (\Xi \hat{F})^T) \delta z \leq 2 \lambda_{\text{max}}((\Xi \hat{F})_{\text{sym}}) \| \delta z \|_2^2 \leq 2 \lambda_{\text{max}}((\Xi \hat{F})_{\text{sym}}) \| \delta z \|_2^2 \tag{36}
\]

where \( (\Xi \hat{F})_{\text{sym}} = [(\Xi \hat{F}) + (\Xi \hat{F})^T] / 2 \). Also, \( \lambda_{\text{max}}((\Xi \hat{F})_{\text{sym}}) > 0 \) and is bounded as \( \lambda_{\text{max}}((\Xi \hat{F})_{\text{sym}}) \leq \min(\lambda_{\text{max}}(K_m), \inf(\lambda_{\text{min}}(K_t^1, \hat{d}_{\text{res,2}}))) \). Hence, it follows from the contraction analysis that all system trajectories converge exponentially fast to a single trajectory at a rate of \( \frac{-\lambda_{\text{max}}((\Xi \hat{F})_{\text{sym}})}{\lambda_{\text{max}}(J_{\text{rot}}^{R,CM})} \), Moreover, in the presence of bounded \( d_{\text{res,2}} \), we get from Lemma 7 in the Appendix that

\[
\lim_{t \to \infty} \int_0^{t} \| \delta \omega_s \|_2 \leq \frac{\lambda_{\text{max}}(J_{\text{rot}}^{R,CM})}{\lambda_{\text{max}}(\Xi \hat{F})_{\text{sym}}} \left( \sup_{I} \lambda_{\text{max}}(K_t^1) \right) (sup_{I} \| \delta_{\text{res,2}} \|_2)
\]

where \( \| \delta \omega_s \|_2 \leq \| \delta z \|_2 \) and \( \lambda_{\text{max}}(J_{\text{rot}}^{R,CM}) > 1 \) are used. Also, note that the disturbance term in the right-hand side of Eq. (35) is \( (0; K_t^{-1} \hat{d}_{\text{res,2}}) \).

Remark 2: Note that the second block diagonal matrix of the Jacobian \( F \) in Eq. (28) is 0, which usually yields a semicontracting system with global asymptotic stability. For example, \( F \) from Eq. (28) and

\[
\Theta = \begin{bmatrix} J_{\text{rot}}^{R,CM} & 0 \\ 0 & I \end{bmatrix}
\]

results in a semicontracting system due to

\[
\frac{1}{2} \left( (\Theta F) + (\Theta F)^T \right) = \begin{bmatrix} -K_m & 0 \\ 0 & 0 \end{bmatrix}
\]

Similarly, the following adaptive control law also yields global asymptotic stability. In contrast, Theorem 2 presents a stronger result with global exponential stability.

D. Nonlinear Adaptive Control

Let the parameter \( \hat{a} \) capture the six uncertain terms in the inertia tensor \( J_{\text{rot}}^{R,CM} \). The resulting adaptive nonlinear tracking control law and the tuning law are given by [20].
\[ u_k = J^{RCM}_{\omega \omega} \dot{\omega} - S(J_{\omega \omega}^{RCM}) \omega - K_\omega (\dot{\omega} - \omega), \]

where \( \omega_k = \ddot{R}^{R}_k \dot{R}_k (\nu(\dot{R}^T_k \dot{R}_k)) - \Lambda, E^T (\dot{R} \, \dot{R}_k) e_k \)

(41)

In the absence of disturbances or uncertainties, this control law guarantees exponential convergence of the system’s trajectory \( R(t) \in SO(3) \) to the desired trajectory \( \dot{R}_d(t) \) for almost all initial conditions, i.e., all initial conditions that are not on the two-dimensional subset of \( SO(3) \) where \( \dot{R} = \dot{R}_d(0) \exp(\pm \pi S(\kappa)) \), where \( \kappa \in \mathbb{S}^2 \). Moreover, in the presence of bounded disturbances or uncertainties, this control law guarantees that if \( \ddot{R}(t) \) will exponentially converge to a bounded error ball around \( \dot{R}_d(t) \).

**Proof:** The closed-loop dynamics obtained by substituting \( u_k \) from Eq. (41) into Eq. (40) is the same as Eq. (14) in the proof of Theorem 1. Hence, we can directly conclude from that proof that all system trajectories of \( \omega_k \) converge exponentially fast to a single trajectory \( \omega_k \rightarrow 0 \) at a rate of \( \lambda_{\min}(E_k, E^T_k) \). Moreover, in the presence of bounded resultant disturbance \( \dot{e}_{res,2} \), \( \lim_{t \to -\infty} \int_0^t \| \| e_k \|_2 \| \) is bounded by Eq. (17).

Now, we show that convergence of \( \omega_k \) implies convergence of the system’s trajectory to the desired trajectory \( e_k \). It follows from the definition of \( \omega_k \) that

\[ \omega_k = \dot{\omega} - \ddot{R}^{R}_k \dot{R}_k (\nu(\dot{R}^T_k \dot{R}_k)) + \Lambda, E^T (\dot{R} \, \dot{R}_k) e_k \]

(42)

In the absence of \( \omega_k \), all system trajectories of \( \omega_k \) will converge exponentially fast to a single trajectory \( \omega_k \rightarrow 0 \) with a rate of \( \lambda_{\min}(E_k, E^T_k) \), where \( E_k, E^T_k \) is also a positive–definite matrix. In the presence of \( \omega_k \), it follows from Lemma 7 in the Appendix that

\[ \lim_{t \to -\infty} \int_0^t \| \| e_k \|_2 \| \leq \frac{\lambda_{\max}(J_{\omega \omega}^{tot})}{\lambda_{\min}(E_k, E^T_k) \lambda_{\min}(J_{\omega \omega}^{tot})} \]

\[ \times (\sup_{t \to -\infty} \| e_k \|_2) \]

(43)

Note that \( \| e_k \|_2 \to \infty \) if \( \ddot{R} \rightarrow \ddot{R}_d \exp(\pm \pi S(\kappa)) \), where \( \kappa \in \mathbb{S}^2 \). On the other hand, for any valid initial condition, \( \| e_k \|_2 \) is always bounded and exponentially decreasing until it reaches the error ball. This implies that once the system starts from a valid initial condition, it can never go toward the two-dimensional subset of \( SO(3) \) due to exponential convergence. Hence, we have shown, using a hierarchical closed-loop system, that the attitude error vector \( e_k \) exponentially converges to the error bound for almost all initial conditions, except for those initial conditions in the two-dimensional subset of \( SO(3) \).

IV. Design of Desired Attitude Trajectory

In this section, we discuss techniques for computing a reference fuel-optimal trajectory and resultant disturbance torque used for the proposed attitude tracking control law in Sec. III. We also outline a framework for minimizing the resultant disturbance torque for the tracking control law.

A. Design of Fuel-Optimal Desired Attitude Trajectory

In this section, we design the desired (reference) attitude trajectory \( q_d(t) \) so that the system reaches the desired attitude orientation \( q_{\text{final}} \) in a fuel-optimal fashion. The original nonlinear optimal control problem is given by

\[ \min_{q_d(t), u_d(t), \omega_d(t)} \int_0^{t_{\text{final}}} \| u_d(t) \|_1 \, dt \]

subject to \( J_{\omega \omega}^{RCM} \omega_d(t) - J_{\omega \omega}^{RCM} \omega_d(t) \times \omega_d(t) - \dot{B} u_d(t) = 0 \)

(45)

where \( \omega_d = \dot{R} \dot{R}_d (\nu(\dot{R}^T_k \dot{R}_d)) - \Lambda, E^T (\dot{R} \, \dot{R}_d) e_k \)

(46)

\[ \| u_d(t) \|_\infty \leq u_{max}, \quad \| \omega_d(t) \|_2 \leq \epsilon_{\text{trans}}, \quad \omega_d(0) = \omega_{init}, \quad \omega_d(t_{\text{final}}) = \omega_{final} \]

(47)

where \( \omega_d(t) \) and \( u_d(t) \) are the fuel-optimal angular velocity and thruster input trajectories. Since all the thrusters generate thrust independently and (there is no gimbaling of thrusters), we use the \( \ell_1 \) vector norm in the \( \ell_1 \) cost function in Eq. (44) [31]. In [14,31–34], a number of optimization strategies are discussed for solving this problem.

We show later in Sec. IV that a relatively negligible amount of fuel is needed for orientating the system to the desired attitude after the angular velocity of the system has stabilized. Therefore, we first find the fuel-optimal angular velocity trajectory that stabilizes the system’s dynamics and control the system using this desired trajectory. The desired fuel-optimal angular velocity trajectory \( \omega_d(t) \) is obtained by solving the following reduced optimal control problem:

\[ \min_{\omega_d(0), \omega_d(t)} \int_0^{t_{\text{final}}} \| u_d(t) \|_1 \, dt \]

(48)

subject to Eqs. (45) and (47).

Because the reduced optimal control problem of \( \omega_d(t) \) in Eq. (48) has fewer optimization constraints than the full optimal control problem of finding both \( q_d(t) \) and \( \omega_d(t) \) in Eq. (44), the solution of the reduced problem in Eq. (48) consumes less fuel than the full problem in Eq. (44). Once \( \omega_d(t) \) is computed from Eq. (48), \( q_d(t) \) is then obtained using the following equations:

\[ \dot{q}_d(t) = Z(q_d(t)) \omega_d(t) \]

\[ \dot{q}_d(t) = \dot{Z}(q_d(t)) \omega_d(t) + Z(q_d(t)) \omega_d(t) \]

(49)

Note that the desired attitude trajectory \( q_d(t) \) obtained using Eq. (49) only stabilizes the angular velocity of the system.

Once the angular velocity of the system is sufficiently close to zero, the desired angular velocity trajectory \( \omega_d(t) \) is augmented with a position error term so that the system’s attitude converges to the desired attitude:

\[ \ddot{\omega}_d(t) = \omega_d(t) - k_{\omega d} Z(q_d(t))^{-1} (q_d(t) - q_{\text{final}}) \]

(50)

where \( k_{\omega d} > 0 \). The desired attitude trajectory \( q_d(t) \) is then obtained from the augmented angular velocity \( \dot{\omega}_d(t) \) using the following equations:

\[ \dot{q}_d(t) = Z(q_d(t)) \dot{\omega}_d(t) = Z(q_d(t)) \omega_d(t) - k_{\omega d} (q_d(t) - q_{\text{final}}) \]

(51)

\[ \dot{q}_d(t) = \frac{d}{dt} [Z(q_d(t)) \dot{\omega}_d(t)] = \dot{Z}(q_d(t)) \omega_d(t) + Z(q_d(t)) \dot{\omega}_d(t) - k_{\omega d} \dot{\omega}_d(t) \]

(52)

These equations are initialized and periodically reset using the current attitude and angular velocity measurements.

B. Desired Attitude Trajectory Using Derivative Plus Proportional–Derivative Control

In this section, we first state the derivative plus proportional–derivative (D+PD) control strategy and then design another desired attitude trajectory \( q_d(t) \) based on the D+PD control strategy.

In the D+PD control strategy, we first use the derivative (rate damping) linear control law for despinning the tumbling system. Once the angular velocity (spin rate) of the system is sufficiently close to zero, the D+PD control strategy switches to a linear proportional–derivative control law to stabilize the attitude of the system in the desired orientation.
Theorem 4: 1) [16,22,24] For the positive–definite symmetric matrix $K_d \in \mathbb{R}^{3 \times 3}$, the derivative (rate damping) control law is given by

$$u_c = -K_d \dot{\omega}$$  \hspace{1cm} (53)

In the absence of disturbances or uncertainties, this control law guarantees global exponential convergence of the system’s angular velocity to 0 rad/s. In the presence of resultant disturbance torque, this control law guarantees that the system’s angular velocity trajectory will globally exponentially converge to a bounded error ball around 0 rad/s.

2) For the positive–definite symmetric matrix $K_d \in \mathbb{R}^{3 \times 3}$ and the constant $k_p > 0$, the proportional–derivative control law is given by

$$u_c = -k_p \beta_{error,v} - K_d \dot{\omega}$$  \hspace{1cm} (54)

where the error quaternion $\beta_{error,v} \in \mathbb{R}^3 \times \mathbb{R}$ represents the orientation error of $F_B$ with respect to the desired target attitude $\beta_{final}$.

This control law only guarantees global asymptotic convergence of the system’s trajectory to the desired trajectory $q_f(t)$ in the absence of disturbances or uncertainties. Hence, the error in the system’s trajectory may not be bounded for a certain class of disturbances [26].

Proof: See [16,22,24,35].

The closed-loop dynamics from Eqs. (53) and (54) are given by

$$J_{tot}^{\beta_{res}} \dot{\omega} - S(J_{tot}^{\beta_{res}} \omega) + K_d \dot{\omega} = d_{res}$$  \hspace{1cm} (55)

$$J_{tot}^{\beta_{res}} \dot{\omega} - S(J_{tot}^{\beta_{res}} \omega) + k_p \beta_{error,v} + K_d \dot{\omega} = d_{res}$$  \hspace{1cm} (56)

It is seen in Sec. II.A.2 that the D+PD control strategy experiences a smaller resultant disturbance torque even in the presence of large $\Delta J_{tot}^{\beta_{res}}$. But the D+PD control strategy does not guarantee global exponential stability (in the absence of disturbances), which is a sufficient condition for satisfying the control problem statement. Hence, we now present the design of a resultant disturbance minimizing desired attitude trajectory for the nonlinear attitude tracking control law Eq. (13).

The desired trajectory is basically broken into two phases. In the first phase, similar to the D+PD control strategy, the desired attitude trajectory is such that $\omega_r = 0$ in Eq. (13) if the magnitude of the system’s angular velocity is large. This ensures that the robust nonlinear tracking control law Eq. (13) effectively reduces to the linear derivative control law Eq. (53) with the same global exponential tracking stability and resultant disturbance torque. In the second phase, once the angular velocity of the system is sufficiently close to zero, we use the following desired attitude trajectory for the nonlinear tracking control law Eq. (13):

$$q_f(t) = q_{final}, \quad \dot{q}_f(t) = 0, \quad : \omega_r = Z^{-1}(\dot{q}), (q_{final} - \dot{q})$$  \hspace{1cm} (57)

Note that this ensures that the system’s attitude globally exponentially converges to the desired final attitude and the system is robust to disturbances. Because the actual angular velocity of the system is small, the resultant disturbance torque is also small even in the presence of large modeling error in $\Delta J_{tot}^{\beta_{res}}$.

The following proposition provides a framework for choosing the desired attitude trajectory so that the resultant disturbance torque $d_{res,2}$ is as small as $d_{res}$.

Proposition 5: Compared with $d_{res}$, the extra terms in $d_{res,2}$ (i.e., $\Delta J_{tot}^{\beta_{res}} \dot{\omega} \times \omega_r$ and $\Delta J_{tot}^{\beta_{res}} \dot{\omega} \times \omega_r$) depend on $\omega_r$, which in turn depends on the desired attitude trajectory. Therefore, the desired attitude trajectory is chosen as follows.

1) If the modeling error in $\Delta J_{tot}^{\beta_{res}}$ is small (i.e., $\| \Delta J_{tot}^{\beta_{res}} \|_2 \leq 10^3$ kg ∙ m²)$^2$, then select the fuel-optimal desired attitude trajectory from Sec. IV.A.

2) Otherwise, select the desired attitude trajectory based on the D+PD control strategy given in Sec. IV.B.

This will ensure that $\| d_{res,2} \|_2 \approx \| d_{res} \|_2$, consequently minimizing the resultant disturbance torque for the robust nonlinear tracking control law Eq. (13).

Proof: Let the worst-case angular velocity of the system be bounded by 0.5 rpm ($\approx 5 \times 10^{-3}$ rad/s), as shown in Table 3. If the fuel-optimal desired trajectory is used, then $\| \omega \|_2 \approx \| \dot{\omega} \|_2$. If the modeling error is small (i.e., $\| \Delta J_{tot}^{\beta_{res}} \|_2 \leq 10^3$ kg ∙ m²), then $\| \Delta J_{tot}^{\beta_{res}} \dot{\omega} \times \omega_r \|_2 \approx 25$ N ∙ m. Neglecting $\omega_r$, which is significantly smaller than $\omega$ and $\dot{\omega}$, we see that $\| d_{res,2} \|_2 \approx \| d_{res} \|_2$.

If the D+PD control strategy-based desired attitude trajectory is used, then $\omega_r = 0$ when $\dot{\omega}$ is large; therefore, $\| \Delta J_{tot}^{\beta_{res}} \dot{\omega} \times \omega_r \|_2 = 0$ N ∙ m and $\| \Delta J_{tot}^{\beta_{res}} \dot{\omega} \|_2 \approx 0$ N ∙ m. If $\dot{\omega}$ is sufficiently close to 0 (i.e., $\| \dot{\omega} \|_2 \leq 5 \times 10^{-3}$ rad/s, and $\| \dot{\omega} \|_2 \approx \| \dot{\omega} \|_2$), and even if the modeling error is large (i.e., $\| \Delta J_{tot}^{\beta_{res}} \|_2 \leq 10^3$ kg ∙ m²), we still get $\| \Delta J_{tot}^{\beta_{res}} \dot{\omega} \times \omega_r \|_2 \leq 2.5$ N ∙ m. Neglecting $\dot{\omega}$ again, we see that $\| d_{res,2} \|_2 \approx \| d_{res} \|_2$.

### Table 3 Simulation parameters (that are same for all simulation cases)

<table>
<thead>
<tr>
<th>Type of parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Spacecraft parameters</strong></td>
<td></td>
</tr>
<tr>
<td>$m_{sc}$</td>
<td>$1.6 \times 10^4$ kg</td>
</tr>
<tr>
<td>$J_{sc}^{\mu} = 10^4 \times \begin{bmatrix} 5.584 &amp; 0 &amp; 0 \ 0 &amp; 5.584 &amp; 0 \ 0 &amp; 0 &amp; 1.568 \end{bmatrix}$</td>
<td>kg ∙ m²</td>
</tr>
<tr>
<td>$r_{sc/IL} = [0 \ 0 \ 3.0]$</td>
<td>m</td>
</tr>
<tr>
<td><strong>Asteroid parameters</strong></td>
<td></td>
</tr>
<tr>
<td>$m_{obj}$</td>
<td>$1.2 \times 10^6$ kg, $r_{obj} = 1.9$ g ∙ cm⁻³</td>
</tr>
<tr>
<td>Shape model: Eros, $J_{obj}^{\mu} = 10^7 \times \begin{bmatrix} 0.8658 &amp; 0.4432 &amp; -0.0005 \ 0.4432 &amp; 3.4900 &amp; 0.0002 \ -0.0005 &amp; 0.0002 &amp; 3.5579 \end{bmatrix}$</td>
<td>kg ∙ m²</td>
</tr>
<tr>
<td>$r_{IL/obj} = [-0.0495 \ -0.0004 \ 3.5456]$</td>
<td>m</td>
</tr>
<tr>
<td><strong>External disturbance actuator error</strong></td>
<td></td>
</tr>
<tr>
<td>$| d_{act} |_2 \approx 1$ N ∙ m, $\Delta \mu = 0$ N</td>
<td></td>
</tr>
<tr>
<td>$q_{initial} = [0.05 \ 0.04 \ 0.03]$</td>
<td></td>
</tr>
<tr>
<td>$\omega_{initial} = [0.01 \ 0.02 \ 0.03]$ rad/s</td>
<td></td>
</tr>
<tr>
<td><strong>Desired final conditions Eqs. (11) and (12)</strong></td>
<td></td>
</tr>
<tr>
<td>$| \omega(t) |<em>2 \leq 0.5$ rpm, $\forall \ t \in \mathbb{R}$, $q</em>{final} = [0 \ 0 \ 0]$, $| q(t) - q_{final} |_2 \leq 10^{-3}$, $\forall \ t &gt; 10^5$ s, $| \omega(t) |_2 \leq 10^{-4}$ rad/s, $\forall \ t &gt; 10^5$ s</td>
<td></td>
</tr>
</tbody>
</table>
V. Simulation Results

In this section, we apply our proposed control law to the ARM attitude control problem of carrying a large unknown object. We first numerically compare the performance of multiple attitude control laws in Sec. VA. We then present a detailed sensitivity analysis of various parameters used in the robust nonlinear tracking control law (Eq. (13)) and the D+PD-based desired attitude trajectory in Sec. IV.B.

A. Comparison of Control Laws for Nonlinear Attitude Control

We use the nominal design of the conceptual ARM spacecraft given in [6] and shown in Fig. 3a. Here, an opposing pair of thrusters in a pod are represented by a single thruster capable of producing thrust between +200 and −200 N. We use the Moore–Penrose pseudoinverse of \( \mathbf{B} \) to allocate thrusts to the eight thrusters in the spacecraft, i.e., \( \mathbf{u} = \mathbf{B}^\dagger (\mathbf{B}^\dagger \mathbf{B})^{-1} \mathbf{u}_r \). Note that we use the right pseudoinverse because the matrix \( \mathbf{B} \) has full row rank, and the matrix inverse \((\mathbf{B}^\dagger \mathbf{B})^{-1}\) is well defined. We do not use the left pseudoinverse because the matrix \((\mathbf{B}^\dagger \mathbf{B})^{-1}\) is usually near singular, and hence its inverse may not be defined.

The fuel consumed by the spacecraft, from time \( t_0 \) to \( t_f \), is computed using the following equation:

\[
\text{Fuel consumed} = \frac{1}{I_sp} \int_{t_0}^{t_f} \| \mathbf{u} \|_1 \, dt
\]  

(58)

where \( I_sp \) is the specific impulse of the fuel (i.e., 287 s for the spacecraft [6]), and \( g_0 \) is the nominal acceleration due to gravity (i.e., 9.8 m/s²).

The shape models of asteroids 433 Eros [37] and 25143 Itokawa [38], shown in Figs. 3b and 3c, are used for generating realistic models of asteroids. We assume that the 16t spacecraft has a captured 1200t asteroid. The objective is to stabilize the rigid spacecraft and spacecraft combination from the given initial conditions to reach the desired final conditions. The simulation parameters, which are the same for all simulation cases, are given in Table 3.

In Table 4, we state the 11 simulation cases considered in this study. These simulation cases are based on varying levels of 1) modeling uncertainties in the estimated inertia tensor of the asteroid \((\Delta J_\text{obj})\), 2) modeling uncertainties in the vector from the spacecraft’s body to the center of mass of the system \((\Delta R_{\text{cm}}/\text{kg m}^2\)), 3) measurement errors in the system’s angular velocity \((\Delta \omega)\), 4) measurement errors in the system’s attitude represented using MRP \((\Delta \mathbf{q})\), and 5) actuator saturations \((u_{\text{max}})\). Each simulation is executed for 1200 s (328 h). The additive measurement errors \((\Delta \omega, \Delta \mathbf{q})\) are simulated using band-limited white noise, where specifies the height of the power spectral density of the white noise, which is the same for each axis. Note that in case 4, the maximum thrust magnitude of each thruster \((u_{\text{max}})\) is increased to 1000 N to avoid actuator saturation.

In this section, we compare the performance of the following attitude control laws: 1) robust nonlinear tracking control law (robust NTCL) [Eq. (13)], 2) adaptive version of the robust nonlinear tracking control law (adaptive RNTCL) [Eq. (38)], and 3) derivative plus proportional–derivative (D+PD) control [Eqs. (53) and (54)]. For the tracking control laws, both the fuel-optimal desired attitude trajectory (Sec. IV.A) and D+PD control-based desired attitude trajectory (Sec. IV.B) are considered. The control law parameters and the parameters for these two desired attitude trajectories are given in Table 5, where the GPOPS-II numerical solver [39] is used to evaluate the fuel-optimal trajectories.

The performances of these control laws for the eleven simulation cases given in Table 4 are shown in Table 6. Some of the notations used in Table 5 are as follows.

\[
\begin{array}{llll}
\text{Case} & \| \Delta \mathbf{J}_\text{obj} \|_2 \cdot 10^6 & \| \Delta R_{\text{cm}}/\text{kg m}^2 \|_2 & \| \Delta \mathbf{q} \|_2 \\
1 & 0 & 0 & 0 \\
2 & 10^3 & 10^{-2} & 0 \\
3 & 10^6 & 10^{-1} & 0 \\
4 & 10^7 & 1 & 0 \\
5 & 0 & 0 & 10^{-12} \\
6 & 0 & 0 & 10^{-10} \\
7 & 10^7 & 1 & 10^{-10} \\
8 & 10^8 & 1 & 10^{-10} \\
9 & 10^9 & 1 & 10^{-10} \\
10 & 10^7 & 1 & 10^{-12} \\
\end{array}
\]

\[
\begin{array}{llll}
\text{Measurement} & \mathcal{P}(\Delta \omega)_2 \text{ rad/s} & \mathcal{P}(\Delta \mathbf{q}) & u_{\text{max}} \text{ N} \\
\text{errors} & 10^{-5} & 10^{-5} & 200 \\
\end{array}
\]

\[
\begin{array}{llll}
\text{Actuator} & \text{saturation} & 10^{-5} & 10^{-5} & 200 \\
\end{array}
\]

\[
\begin{array}{llll}
\text{Case} & \| \Delta \mathbf{J}_\text{obj} \|_2 \cdot 10^6 & \| \Delta R_{\text{cm}}/\text{kg m}^2 \|_2 & \| \Delta \mathbf{q} \|_2 \\
1 & 0 & 0 & 0 \\
2 & 10^3 & 10^{-2} & 0 \\
3 & 10^6 & 10^{-1} & 0 \\
4 & 10^7 & 1 & 0 \\
5 & 0 & 0 & 10^{-12} \\
6 & 0 & 0 & 10^{-10} \\
7 & 10^7 & 1 & 10^{-10} \\
8 & 10^8 & 1 & 10^{-10} \\
9 & 10^9 & 1 & 10^{-10} \\
10 & 10^7 & 1 & 10^{-12} \\
\end{array}
\]

\[
\begin{array}{llll}
\text{Measurement} & \mathcal{P}(\Delta \omega)_2 \text{ rad/s} & \mathcal{P}(\Delta \mathbf{q}) & u_{\text{max}} \text{ N} \\
\text{errors} & 10^{-5} & 10^{-5} & 200 \\
\end{array}
\]

\[
\begin{array}{llll}
\text{Actuator} & \text{saturation} & 10^{-5} & 10^{-5} & 200 \\
\end{array}
\]
1) The angular velocity convergence time $t_{\omega, \text{conv}}$ denotes the least time instant after which the system’s angular velocity $\omega(t)$ is always below the given threshold of $10^{-2}$ rad/s (i.e., $\|\omega(t)\|_2 \leq 10^{-4}$ rad/s, $\forall t > t_{\omega, \text{conv}}$).

2) The attitude convergence time $t_{\mathbf{q}, \text{conv}}$ denotes the least time instant after which the error in the system’s attitude $\|\mathbf{q}(t) - \mathbf{q}_{\text{final}}\|$ is always below the given threshold of $10^{-2}$ (i.e., $\|\mathbf{q}(t) - \mathbf{q}_{\text{final}}\|_2 \leq 10^{-2}$, $\forall t > t_{\mathbf{q}, \text{conv}}$). Note that after time $t_{\mathbf{q}, \text{conv}}$, the attitude control law can be switched off because the asteroid and spacecraft combination has been three-axis stabilized in the final desired orientation. The fuel consumed up to time $t_{\omega, \text{conv}}$ and $t_{\mathbf{q}, \text{conv}}$ are also shown in Table 6.

3) The symbol $\mathcal{N}C$ ("not converged") refers to the case when the control law is not able to stabilize the system due to actuator saturation. In the absence of measurement errors and modeling uncertainties (case 1), Figs. 4a–4c show a result of the nonlinear tracking control law (robust NTCL) tracking the fuel-optimum reference trajectory. Note that Fig. 4c also shows the fuel consumption level for the case where the fuel-optimal $\omega_d(t)$ trajectory is not augmented (i.e., $k_qd = 0$) and consequently only the angular velocity of the system converges. We can infer from this plot that a relatively negligible amount of fuel ($\approx 3$ kg) is used for stabilizing the attitude of the asteroid and spacecraft combination using the augmented angular velocity $\omega_d(t)$ in Eq. (50).

Table 6 Performance of control laws for the simulation cases in Table 4

<table>
<thead>
<tr>
<th>Convergence time and fuel consumed</th>
<th>Robust NTCL [Eq. (13)]</th>
<th>Adaptive RNTCL [Eq. (38)]</th>
<th>D+PD-based trajectory</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t_{\omega, \text{conv}}$, s</td>
<td>5.23 x 10^4</td>
<td>5.32 x 10^4</td>
<td>2.34 x 10^4</td>
</tr>
<tr>
<td>Fuel at $t_{\omega, \text{conv}}$, kg</td>
<td>82.5</td>
<td>106.4</td>
<td>135.2</td>
</tr>
<tr>
<td>$t_{\mathbf{q}, \text{conv}}$, s</td>
<td>8.57 x 10^4</td>
<td>8.56 x 10^4</td>
<td>3.01 x 10^4</td>
</tr>
<tr>
<td>Fuel at $t_{\mathbf{q}, \text{conv}}$, kg</td>
<td>82.9</td>
<td>106.9</td>
<td>135.5</td>
</tr>
<tr>
<td>$t_{\omega, \text{conv}}$, s</td>
<td>5.23 x 10^4</td>
<td>5.32 x 10^4</td>
<td>2.35 x 10^4</td>
</tr>
<tr>
<td>Fuel at $t_{\omega, \text{conv}}$, kg</td>
<td>82.7</td>
<td>113.0</td>
<td>134.3</td>
</tr>
<tr>
<td>$t_{\mathbf{q}, \text{conv}}$, s</td>
<td>8.57 x 10^4</td>
<td>8.57 x 10^4</td>
<td>3.03 x 10^4</td>
</tr>
<tr>
<td>Fuel at $t_{\mathbf{q}, \text{conv}}$, kg</td>
<td>83.1</td>
<td>113.4</td>
<td>134.5</td>
</tr>
<tr>
<td>$t_{\omega, \text{conv}}$, s</td>
<td>2.30 x 10^4</td>
<td>2.37 x 10^4</td>
<td>2.38 x 10^4</td>
</tr>
<tr>
<td>Fuel at $t_{\omega, \text{conv}}$, kg</td>
<td>138.9</td>
<td>130.2</td>
<td>127.9</td>
</tr>
<tr>
<td>$t_{\mathbf{q}, \text{conv}}$, s</td>
<td>5.56 x 10^4</td>
<td>5.59 x 10^4</td>
<td>3.07 x 10^4</td>
</tr>
<tr>
<td>Fuel at $t_{\mathbf{q}, \text{conv}}$, kg</td>
<td>139.4</td>
<td>130.8</td>
<td>128.2</td>
</tr>
<tr>
<td>$t_{\omega, \text{conv}}$, s</td>
<td>2.86 x 10^4</td>
<td>3.06 x 10^4</td>
<td>2.07 x 10^4</td>
</tr>
<tr>
<td>Fuel at $t_{\omega, \text{conv}}$, kg</td>
<td>1220.1</td>
<td>814.7</td>
<td>115.1</td>
</tr>
<tr>
<td>$t_{\mathbf{q}, \text{conv}}$, s</td>
<td>6.04 x 10^4</td>
<td>6.14 x 10^4</td>
<td>2.61 x 10^4</td>
</tr>
<tr>
<td>Fuel at $t_{\mathbf{q}, \text{conv}}$, kg</td>
<td>1220.3</td>
<td>814.9</td>
<td>115.6</td>
</tr>
<tr>
<td>$t_{\omega, \text{conv}}$, s</td>
<td>5.32 x 10^4</td>
<td>5.31 x 10^4</td>
<td>2.34 x 10^4</td>
</tr>
<tr>
<td>Fuel at $t_{\omega, \text{conv}}$, kg</td>
<td>83.2</td>
<td>86.0</td>
<td>135.3</td>
</tr>
<tr>
<td>$t_{\mathbf{q}, \text{conv}}$, s</td>
<td>8.60 x 10^4</td>
<td>8.51 x 10^4</td>
<td>3.01 x 10^4</td>
</tr>
<tr>
<td>Fuel at $t_{\mathbf{q}, \text{conv}}$, kg</td>
<td>83.6</td>
<td>86.4</td>
<td>135.5</td>
</tr>
<tr>
<td>$t_{\omega, \text{conv}}$, s</td>
<td>5.32 x 10^4</td>
<td>5.31 x 10^4</td>
<td>2.34 x 10^4</td>
</tr>
<tr>
<td>Fuel at $t_{\omega, \text{conv}}$, kg</td>
<td>83.1</td>
<td>87.5</td>
<td>135.2</td>
</tr>
<tr>
<td>$t_{\mathbf{q}, \text{conv}}$, s</td>
<td>8.56 x 10^4</td>
<td>8.48 x 10^4</td>
<td>3.01 x 10^4</td>
</tr>
<tr>
<td>Fuel at $t_{\mathbf{q}, \text{conv}}$, kg</td>
<td>83.7</td>
<td>88.9</td>
<td>135.5</td>
</tr>
<tr>
<td>$t_{\omega, \text{conv}}$, s</td>
<td>$\mathcal{N}C$</td>
<td>$\mathcal{N}C$</td>
<td>2.05 x 10^4</td>
</tr>
<tr>
<td>Fuel at $t_{\omega, \text{conv}}$, kg</td>
<td>$\mathcal{N}C$</td>
<td>$\mathcal{N}C$</td>
<td>2.05 x 10^4</td>
</tr>
<tr>
<td>$t_{\mathbf{q}, \text{conv}}$, s</td>
<td>$\mathcal{N}C$</td>
<td>$\mathcal{N}C$</td>
<td>2.59 x 10^4</td>
</tr>
<tr>
<td>Fuel at $t_{\mathbf{q}, \text{conv}}$, kg</td>
<td>$\mathcal{N}C$</td>
<td>$\mathcal{N}C$</td>
<td>2.59 x 10^4</td>
</tr>
<tr>
<td>$t_{\omega, \text{conv}}$, s</td>
<td>2.08 x 10^4</td>
<td>2.08 x 10^4</td>
<td>2.08 x 10^4</td>
</tr>
<tr>
<td>Fuel at $t_{\omega, \text{conv}}$, kg</td>
<td>$\mathcal{N}C$</td>
<td>$\mathcal{N}C$</td>
<td>2.81 x 10^4</td>
</tr>
<tr>
<td>$t_{\mathbf{q}, \text{conv}}$, s</td>
<td>2.36 x 10^4</td>
<td>2.36 x 10^4</td>
<td>2.74 x 10^4</td>
</tr>
<tr>
<td>Fuel at $t_{\mathbf{q}, \text{conv}}$, kg</td>
<td>$\mathcal{N}C$</td>
<td>$\mathcal{N}C$</td>
<td>2.77 x 10^4</td>
</tr>
<tr>
<td>$t_{\omega, \text{conv}}$, s</td>
<td>2.05 x 10^4</td>
<td>2.05 x 10^4</td>
<td>2.05 x 10^4</td>
</tr>
<tr>
<td>Fuel at $t_{\omega, \text{conv}}$, kg</td>
<td>$\mathcal{N}C$</td>
<td>$\mathcal{N}C$</td>
<td>2.59 x 10^4</td>
</tr>
<tr>
<td>$t_{\mathbf{q}, \text{conv}}$, s</td>
<td>2.05 x 10^4</td>
<td>2.05 x 10^4</td>
<td>2.59 x 10^4</td>
</tr>
<tr>
<td>Fuel at $t_{\mathbf{q}, \text{conv}}$, kg</td>
<td>$\mathcal{N}C$</td>
<td>$\mathcal{N}C$</td>
<td>2.59 x 10^4</td>
</tr>
</tbody>
</table>
We conclude from cases 1 and 2 that, in the absence of measurement errors and under smaller modeling uncertainties, which can be achieved using online system identification techniques, the method of robust NTCL tracking the fuel optimal trajectory is the best strategy because it guarantees exponential convergence to the desired trajectory, thereby consuming the least fuel. One caveat of using this control law is that the reduced nonlinear optimal control problem in Eq. (48) should be solved in real time for the given initial angular velocity \( \omega_{\text{initial}} \) and the estimated inertia tensor of the combined system \( \mathbf{I}_{\text{CM}} \).

Cases 3 and 4 show that the nonlinear tracking control laws, which track the fuel-optimal desired attitude trajectory, consume more fuel than the same nonlinear tracking control laws that track the D+PD control strategy-based desired attitude trajectory because of the large resultant disturbance torque discussed in Sec. II.A.2. Note that the angular velocity convergence times \( (t_{\text{conv}}) \) of the nonlinear control laws for cases 1–4 are different because different values of \( \mathbf{J}_{\text{CM}} \) are used in the nonlinear control optimal problem [Eq. (48)] to obtain the fuel-optimal desired attitude trajectories \( [\omega(t), q(t)] \) given in Sec. IV.A.

In cases 5–11, a simple filtering algorithm is used to remove the additive noise from the measured states. In this filtering algorithm, the states \( (\omega, q) \) are first predicted using the nonlinear dynamics and kinematics equations as well as state values from the previous time instant. Then, the errors between the measured states and the predicted states are filtered using a low-pass filter, a first-order filter with transfer function \( (\omega_{\text{coarse}})(s) + \omega_{\text{coarse}} \), where \( \omega_{\text{coarse}} = 0.02\pi \text{ rad/s} \), to remove the high-frequency components arising from the noise. Finally, the filtered errors are added to the predicted states to retrieve the estimated states \( (\hat{\omega}, \hat{q}) \).

Cases 5 and 6 show that the robust NTCL consumes less fuel while tracking the fuel-optimal trajectory compared to the D+PD control strategy-based trajectory in the presence of small measurement errors. In contrast, case 7 shows that this robust NTCL cannot stabilize the system in the presence of both large modeling errors and measurement errors because of actuator saturation caused by the large resultant disturbance torque discussed in Sec. II.A.2.

If we use the resultant disturbance torque-time minimizing, D+PD control strategy-based desired attitude trajectory, then the robust NTCL can stabilize the system in the presence of both large measurement errors and large modeling errors. Moreover, the fuel consumed and the time of convergence do not change much with uncertainties and errors, as seen in cases 1–7 in Table 6. Moreover, case 7 shows the worst-case measurement errors for the desired convergence bounds because if the measurement errors (noise levels) increase above these values, then the instantaneous magnitude of the measurement errors become comparable to the desired convergence bounds in Table 3, and the spacecraft expends fuel continuously to counter these errors. Therefore, these uncertainty and error limits determine the required technical capabilities of the sensors and actuators onboard the spacecraft.

The simulation results (trajectories) of the robust NTCL for case 7 are shown in Figs. 4d–4f. Note that the net fuel consumed \((\approx 120 \text{ kg})\) after \(10^5 \text{ s} \) is comfortably within the fuel capacity of the spacecraft \((i.e., 900 \text{ kg} \text{ [4]})\). Figure 4f also shows the fuel consumption for the case where only the derivative (rate damping) control law Eq. (53) is used for the entire time, and consequently only the angular velocity of the system converges. We can infer from this plot that a comparatively negligible amount of fuel \((\approx 5 \text{ kg})\) is used by the proportional term in Eq. (57) for stabilizing the attitude of the asteroid and spacecraft combination.

Case 7 also gives the worst-case modeling errors because \( \| \Delta \mathbf{J}_{\text{CM}} \| \approx \| \mathbf{J}_{\text{CM}} \| \text{ and } \| \Delta \mathbf{B}_{\text{CM}} \| \approx \| \mathbf{B}_{\text{CM}} \| \text{ in cases } 8–11, \) we study the effect of each of these uncertainties by individually reducing them from their worst-case bounds. Note that the control laws tracking the fuel-optimal desired trajectory are unable to stabilize the system because of actuator saturation. The robust NTCL, which tracks the D+PD-based reference trajectory, gives satisfactory performance for these cases too.

B. Sensitivity Analysis of the Robust Nonlinear Tracking Control Law

We now present detailed sensitivity analysis of the robust NTCL and the D+PD control strategy-based desired attitude trajectory, by varying the asteroid parameters, the initial conditions, and the control law parameters. The parameters that are not explicitly specified are taken from Tables 3 and 5 and from case 7 in Table 4. Figure 5 shows the variation of the convergence time of the angular velocity \( (t_{\text{conv}}) \), the convergence time of the attitude \( (t_{\text{conv}}) \), and the fuel consumed up to time \( t_{\text{conv}} \) with respect to the mass and density of the model asteroid. The inset white trapezium in Fig. 5 shows the nominal range of NEO asteroid parameters \((i.e., \text{ the asteroid's mass is within } 2.5–13 \times 10^3 \text{ kg}, \text{ the asteroid's density is within} \)
Itokawa respectively.

Fig. 6 Sensitivity plots show the effect of the damping gain $K_r$ on the convergence times and fuel consumption for the two shape models of Eros and Itokawa, respectively.

The effect of this damping gain, which is given by $K_r \approx 10^4$, on the fuel consumption and the convergence time is shown in Fig. 6. Even though $K_r$ is varied from $0.5 - 2.5 \times 10^4$, its effect on the fuel consumed to stabilize the system is minimal, as shown in Figs. 6a and 6d. On the other hand, increasing $K_r$ reduces the convergence time of the angular velocity, as shown in Figs. 6b and 6e, and the convergence time of the attitude, as shown in Figs. 6c and 6f. If $K_r$ is chosen to be less than $0.5 \times 10^4$, then the angular velocity convergence time increases beyond $10^2$ s, which is not desirable. If $K_r$ is chosen to be greater than $2.5 \times 10^4$, then the system converges quickly, but the control action becomes very sensitive to angular velocity measurement errors. Hence, the damping gain of $K_r \approx 10^4$ is ideal for this mission.
The effect of the tuning parameter, which is given by $\Lambda_r = \lambda_r I$, on the fuel consumption and the convergence time is shown in Fig. 7.

As expected, its effect is minimal because the tuning parameter is only used after the angular velocity of the system is sufficiently close to zero in the D+PD-based reference trajectory. Therefore, we recommend using any tuning parameter within the range of $0.5 - 2.5 \times 10^{-3}$.

It is shown in Figs. 5a and 5d that the asteroid with mass $m_{obj} = 1.1 \times 10^6$ kg and density $\rho_{obj} = 1.9$ g cm$^{-3}$ consumes the maximum fuel among all asteroids in the nominal range. The effect of the initial angular velocity $\omega_{initial}$ on the fuel consumption and convergence time for this nominal asteroid is shown in Fig. 8, where all initial angular velocities are on the sphere with radius of 0.5 rpm. We observe that the fuel consumed by the spacecraft-asteroid system is less than 15 m).

Fig. 7 Sensitivity plots show the effect of the tuning parameter $\Lambda_r = \lambda_r I$ on the convergence times and fuel consumption for the two shape models of Eros and Itokawa respectively (asteroid’s diameter is less than 15 m).

Fig. 8 Sensitivity plots show the effect of the initial angular velocity $\omega_{initial}$ on the convergence time and the fuel consumed for the two shape models of Eros and Itokawa respectively, where $m_{obj} = 1.1 \times 10^6$ kg and $\rho_{obj} = 1.9$ g cm$^{-3}$. 
combination is upper bounded by 300 kg for all initial conditions, as shown in Figs. 8a and 8c. Hence, we conclude that the fuel consumed by the spacecraft–asteroid combination using the robust NTCL, Eq. (13), which tracks the D+PD control strategy–based desired attitude trajectory (Sec. IV.B), is upper bounded by 300 kg for the nominal range of asteroid parameters (i.e., the asteroid’s mass is within 2.5–13 × 10^3 kg, the asteroid’s density is within 1.9–3.8 g · cm^−3, and the asteroid’s diameter is less than 15 m). Note that the convergence times of the angular velocities and the attitudes are satisfactory for all initial conditions. Moreover, the effect of the initial attitude change on the fuel consumption and convergence time is negligible.

VI. Conclusions

A new robust nonlinear tracking control law for attitude control of a spacecraft with large uncertainty is presented, which guarantees both global exponential convergence to the desired attitude trajectory and bounded tracking errors (in the sense of finite–gain L∞ stability and ISS) in the presence of uncertainties and disturbances. The benefits of this new attitude tracking control law include superior robustness due to no feedforward cancellation and straightforward extensions to integral control and various attitude representations such as MRPs and SO(3). A comparison of the resultant disturbance torques produced by various types of attitude control laws is presented, and it is concluded that the proposed control law could produce a small resultant disturbance torque if the desired trajectory was designed appropriately. Techniques were also discussed for obtaining fuel-optimal or resultant disturbance torque minimizing desired attitude trajectories for these nonlinear tracking control laws.

The performance of multiple control laws was then numerically compared, such as the proposed robust nonlinear tracking control law, nonlinear adaptive control, and the D+PD linear control strategy, for a spacecraft–asteroid combination with large modelling uncertainty. It was illustrated that, in the presence of small measurement errors and small modeling uncertainties, which could be achieved using online system identification, the robust nonlinear tracking control law that tracked a fuel–optimal reference trajectory was the best strategy because it consumed the least amount of fuel. It was also shown that a comparatively negligible amount of fuel was needed for the combined system to the desired orientation after the angular velocity of the system was stabilized. One caveat of using both optimal control and nonlinear tracking control is that the spacecraft should have sufficient computational power for online system identification and real-time-fuel-optimal trajectory generation.

On the other hand, in the presence of large modeling uncertainties, measurement errors, and actuator saturations, or in the absence of sufficient computational power onboard the spacecraft, the simple linear D+PD control strategy resulted in good performance. This performance was further enhanced with properties of superior robustness and tracking convergence if the robust nonlinear tracking control law was used to globally exponentially track a desired attitude trajectory that was generated using the D+PD linear control strategy. It is envisaged that the design guidelines presented in this paper can be useful for a future asteroid capture or redirection mission.

Appendix: Contraction Theory

In this paper, we use contraction theory to prove the stability of control laws. In this section, we present some results on contraction theory from [28,40]. Readers are referred to these references for detailed descriptions and proofs for the following theorems.

Lemma 6 (Contraction analysis [40]): We consider a smooth nonlinear nonautonomous system:

\[ \dot{x}(t) = f(x(t), t), \quad x(t) \in \mathbb{R}^n \]  

(A1)

A virtual displacement \( \delta x \) is defined as an infinitesimal displacement at fixed time, and \( \Theta(x, t) \) is a smooth coordinate transformation of the virtual displacement such that \( \delta z = \Theta \delta x \). Then, if there exists a positive \( \lambda \) and a uniformly positive-definite metric \( M(x, t) = \Theta(x, t)^T \Theta(x, t) \), such that

\[
\frac{d}{dt}(\delta z^T \delta z) = \frac{d}{dt}(\delta x^T M(x, t) \delta x) = \delta x^T \left( \dot{M} + \left( \frac{\partial f}{\partial x} \right)^T M + M \frac{\partial f}{\partial x} \right) \delta x \leq -2\lambda \delta x^T M(x, t) \delta x \tag{A2}
\]

then all system trajectories converge exponentially fast to a single trajectory regardless of the initial conditions \( (\delta z, \delta x) \rightarrow 0 \) at a rate of \( \lambda \) (i.e., contracting), and \( \lambda \) is the largest eigenvalue of the symmetric part of

\[ \Theta + \Theta \frac{\partial f}{\partial x} \Theta^{-1} \]

We define the \( L_p \) norm in the extended space \( L_{pe}, p \in [1, \infty) \) as follows [28]:

\[
\|(u(t))\|_{L_p} = \left( \int_0^\tau \left\| u(t) \right\|_p^p \, dt \right)^{1/p} < \infty, \quad p \in [1, \infty), \tag{A3}
\]

\[
\|(u(t))\|_{L_\infty} = \sup_{t \geq 0} \|u(t)\|_2 < \infty \]

where \((u(t))_i \) is a truncation of \( u(t) \), i.e., \((u(t))_i = 0 \) for \( i \geq \tau, \tau \in [0, \infty) \) whereas \((u(t))_i = u(t) \) for \( 0 \leq i \leq \tau \).

Lemma 7 (Robust contraction and link to \( L_p \) stability and ISS [28]): Let \( P_1(t) \) be a solution of the contracting system [Eq. (A1)] globally exponentially tending to a single trajectory at a contraction rate of \( \lambda \). Equation (A1) is now perturbed as

\[ \dot{x}(t) = f(x(t), t) + d(x(t), t) \quad \tag{A4} \]

and \( P_2(t) \) denotes the trajectory of Eq. (A4). Then, the smallest path integral (i.e., distance) \( R(t) = \int_0^\tau \| \Theta(x, t) \dot{x}(t) \|_2 \, dt \), \( \forall \tau \geq 0 \) exponentially converges to the following error ball [40]:

\[
\lim_{t \to \infty} R(t) \leq \sup_{x, \lambda} \left\| \Theta(x, t) d(x(t), t) \right\|_2 \leq \frac{\lambda}{\lambda - \eta} \quad \tag{A5}
\]

with \( \lambda \) \text{ } \in L_{\infty}. \) Furthermore, if \( d(x(t), t) \in L_{pe} \), then Eq. (A4) is finite-gain \( L_p \) stable with \( p \in [1, \infty) \) for an output function \( y = h(x(t), d(t)) \) with \( \int_{t_1}^{t_2} \left\| \frac{\partial y}{\partial x} \right\|_2 \leq \eta_1 \int_{t_1}^{t_2} \left\| \dot{x}(t) \right\|_2 + \eta_2 \left\| d(t) \right\|_2, \exists \eta_1, \eta_2 \geq 0 \), because

\[
\left\| \left( \int_{t_1}^{t_2} \frac{\partial y}{\partial x} \right)_i \right\|_{L_p} \leq \frac{\eta_1 \zeta R(0)}{\lambda_{\min}^p(M)} + \left( 1 + \frac{\lambda}{\lambda - \eta} \right) \left( \frac{\|\Theta(x)\|_{L_p}}{\sqrt{\lambda_{\min}(M)}} \right). \quad \forall \tau \in [0, \infty) \tag{A6}
\]

where \( Y_1(t) \) and \( Y_2(t) \) denote the output trajectories of the original contracting system and its perturbed system, respectively, and \( \zeta = 1 \) if \( p = \infty \) or \( \zeta = 1/(p-1) \) if \( p \in [1, \infty) \). The perturbed system [Eq. (A4)] is also input-to-state stable.

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References

