Probabilistic and Distributed Control of a Large-Scale Swarm of Autonomous Agents

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Abstract—We present a novel method for guiding a large-scale swarm of autonomous agents into a desired formation shape in a distributed and scalable manner. Our Probabilistic Swarm Guidance using Inhomogeneous Markov Chains (PSG–IMC) algorithm adopts an Eulerian framework, where the physical space is partitioned into bins and the swarm’s density distribution over each bin is controlled. Each agent determines its bin transition probabilities using a time-inhomogeneous Markov chain. These time-varying Markov matrices are constructed by each agent in real-time using the feedback from the current swarm distribution, which is estimated in a distributed manner. The PSG–IMC algorithm minimizes the expected cost of the transitions per time instant, required to achieve and maintain the desired formation shape, even when agents are added to or removed from the swarm. The algorithm scales well with a large number of agents and complex formation shapes, and can also be adapted for area exploration applications. We demonstrate the effectiveness of this proposed swarm guidance algorithm by using results of numerical simulations and hardware experiments with multiple quadrotors.

Index Terms—swarm robotics, multi-agent systems, Markov chains, path planning, guidance.

I. INTRODUCTION

A large-scale swarm of space robotic systems could collaboratively complete tasks that are very difficult for a single agent, with significantly enhanced flexibility, adaptability, and robustness [1]. Moreover, a large-scale swarm (having $10^3$–$10^6$ or more agents) may be deployed for challenging missions like constructing a complex formation shape (see Fig. 1) [2]–[8] or exploring an unknown environment [9]–[14]. The control algorithm for such a large-scale swarm of autonomous agents should be:

- Distributed: The algorithm should not need a centralized supervisor or controller.
- Versatile: The algorithm can be easily tailored for multiple applications such as maintaining the formation shape or exploring the target area.
- Robust: Since a small fraction of agents in the swarm might get lost during the course of an operation or new agents might get added to the swarm, the algorithm should seamlessly adapt to loss or addition of agents. Moreover, the algorithm should accommodate measurement errors, actuation errors, and other uncertainties.

In this paper, we lay the theoretical foundations of a distributed, versatile, robust, and scalable algorithm for controlling the shape of large-scale swarms.

In fluid mechanics, the motion of fluids is described using either a Lagrangian framework, where each fluid particle’s motion is tracked; or an Eulerian framework, where the fluid’s density distribution at specific locations is tracked. Analogously, distributed control algorithms for swarms can be classified into two categories [15], [16]: the individual-agent-based Lagrangian framework, where each agent’s trajectory is generated separately [2]–[14]; and the continuum-based Eulerian framework, where the collective properties of the swarm (e.g., its density distribution) is controlled. In the Lagrangian framework, the computation cost for generating each agent’s trajectory increases rapidly with a large number ($10^3$–$10^6$ or more) of agents. For example, the computational complexity of each agent’s target assignment problem increases at least quadratically with the number of agents [5]–[8]. The Eulerian framework decouples these two tasks by first solving the target assignment problem at a coarser spatial resolution. Moreover, the Eulerian framework does not efficiently handle loss or addition of agents, nor does it scale well with the size of the area and arbitrary formation shapes [2]–[4]. Consequently, we adopt the Eulerian framework in this paper.

A. Literature Review

Numerous path planning algorithms within the Lagrangian framework are discussed in the survey papers on swarm robotics [17], [18]. In this subsection, we focus on guidance...
algorithms that use an Eulerian approach [19]–[21]. For shape formation and reconfiguration applications, the physical space over which the swarm is distributed is first tessellated (i.e., partitioned) into discrete bins [22], [23]. The bin size is determined by the spatial resolution of the desired formation shape. Assuming that the number of agents is much larger than the number of non-empty bins, the density distribution of the swarm over these bins is controlled to achieve the desired formation shape.

Within the Eulerian framework, Homogeneous Markov Chain (HMC) based algorithms are a popular choice for shape formation [24]–[27], area exploration [28], [29], task allocation [30], [31], and surveillance applications [32], [33]. In such algorithms, the agent’s transition probability between bins is encoded in a constant Markov matrix that has the desired formation shape as its stationary distribution. Such an approach is probabilistic, as opposed to deterministic, because each agent determines its next bin location by inverse-transform sampling of the Markov matrix [34]. Since the number of agents in the swarm can vary with time and the agents do not keep track of the time-varying number of agents in the swarm, a probabilistic approach is preferred because a deterministic target assignment algorithm needs to keep track of the changes in the number of agents and targets [7]. Moreover, as shall be seen in the paper, our probabilistic approach can also handle measurement uncertainties and actuation errors in a robust manner. The HMC-based algorithms possess the aforementioned benefits of robustness and scalability, because addition or removal of agents from the swarm does not affect the convergence of the HMC to the stationary distribution.

However, the major drawback of these HMC-based algorithms is that they are inherently open-loop strategies which cannot incorporate any feedback. Clearly, the effectiveness of these algorithms can be greatly improved by refining the Markov matrix at each time step using some feedback. Such refinement results in an Inhomogeneous Markov Chain (IMC), which is at the core of our algorithm.

In this paper, we derive the Probabilistic Swarm Guidance using Inhomogeneous Markov Chains (PSG–IMC) algorithm, which incorporates the feedback from the current swarm density distribution at each time step. The PSG–IMC algorithm is a closed-loop distributed guidance strategy that retains the original robustness and scalability properties associated with a Markovian approach. Another disadvantage of HMC-based algorithms is that they suffer undesirable transitions, i.e., transitions from bins that are deficient in agents to bins with surplus agents. Such undesirable transitions prevent the swarm from converging to the desired formation. The PSG–IMC algorithm suppresses such undesirable transitions between bins, thereby reducing the control effort needed for achieving and maintaining the formation. This benefit also results in smaller convergence errors than HMC-based algorithms.

The motion of swarm agents can be formulated as an optimal transport problem with respect to a given cost function [35]. If perfect feedback of the current swarm distribution is available, then our prior work on the Probabilistic Swarm Guidance using Optimal Transport (PSG–OT) algorithm [36] gives good performance. However, there are two major dis-advantages of such an approach. First, the performance of an optimal transport-based algorithm drops precipitously with estimation errors in the feedback loop. Measurement and estimation errors are routinely encountered in practice and it is often impossible or impractical to generate perfect feedback of the current swarm distribution. Second, the computation time of the optimization problem increases very fast with an increasing number of bins. This is a notable drawback because a large number of bins are necessary for capturing fine spatial details in the desired formation shape. The PSG–IMC algorithm proposed in the present paper can overcome both challenges, since it works effectively in the presence of error-prone feedback and scales well with a large number of bins.

A different approach to swarm formation within the Eulerian framework is to model the continuum dynamics using partial differential equations (PDEs) [37]–[39]. Since our goal is to achieve arbitrary formation shapes that are not limited to the equilibrium states of the PDE, we do not consider a PDE-based approach. Our approach is also different from the multiagent Markov decision process approach in [40], [41] because our agents do not keep track of the states and actions of other agents.

Distributed Eulerian approaches for area exploration applications using region-based shape controllers and attraction-repulsion forcing functions are discussed in [42]–[44]. We show that a slight modification of our PSG–IMC algorithm also results in an efficient area-exploration algorithm.

B. Main Contributions

The first contribution of this paper is a novel technique for constructing feedback-based Markov matrices for a given stationary distribution which represents the desired formation, where the expected cost of transitions at each time instant is minimized. Each Markov matrix satisfies the motion constraints that might arise due to the dynamics or other physical constraints. The Markov matrix converges to the identity matrix when the swarm converges to the desired formation, thereby ensuring that the swarm settles down and reducing unnecessary transitions.

Second, we describe the PSG–IMC algorithm for shape formation. We rigorously derive the convergence proofs for the PSG–IMC algorithm based on the analysis of IMC, which is more involved than the convergence proof for HMC. We show that each agent’s IMC strongly ergodically converges to the desired formation shape. Further, we provide a time-varying probabilistic-bound on the convergence error as well as a lower bound on the number of agents for ensuring that the final convergence error is below the desired threshold. Furthermore, we present an extension of the PSG–IMC algorithm for area exploration applications.

Along with a lower-level collision-free motion planner, we demonstrate using multiple quadrotors that the PSG–IMC algorithm can be executed in real-time to achieve multiple desired formation shapes. Using results of numerical simulations, we show that the PSG–IMC algorithm yields a smaller convergence error and more robust convergence than the HMC-based and PSG–OT algorithms, while significantly
reducing the number of transitions in the presence of estimation errors. Thus, the PSG–IMC algorithm is best-suited for large-scale swarms with error-prone feedback and complex desired formations with a large number of bins.

Compared to our conference paper [45], we have added detailed proofs in the convergence analysis section and extensions of our algorithm for shape formation and area exploration applications. We have also added extensive numerical and experimental results to this paper.

This paper is organized as follows. The problem statement for shape formation is discussed in Section II. In Section III, we present our novel techniques for constructing Markov matrices. The PSG–IMC algorithm for shape formation is presented in Section IV. The PSG–IMC algorithm for area exploration is presented in Section V. Results of numerical simulations and experimentation are presented in Section VI. This paper is concluded in Section VII.

Table I

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Explanation</th>
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<tbody>
<tr>
<td>$A_i^j$</td>
<td>Motion constraints matrix of the $j^{th}$ agent (Definition 4)</td>
</tr>
<tr>
<td>$B[i]$</td>
<td>Bins, where $1 \leq i \leq n_{bin}$ (Definition 1)</td>
</tr>
<tr>
<td>$M_k^j$</td>
<td>Markov matrix of the $j^{th}$ agent (6)</td>
</tr>
<tr>
<td>$S_k^j$</td>
<td>Condition for escaping transient bins (17)</td>
</tr>
<tr>
<td>$C_k$</td>
<td>Cost matrix at the $k^{th}$ time instant (Definition 5)</td>
</tr>
<tr>
<td>$m_k$</td>
<td>Number of agents in the swarm (Assumption 1)</td>
</tr>
<tr>
<td>$r_k^j$</td>
<td>Actual bin position of the $j^{th}$ agent (Assumption 3)</td>
</tr>
<tr>
<td>$x_k^j$</td>
<td>Probability vector of the $j^{th}$ agent predicted position (22)</td>
</tr>
<tr>
<td>$\Theta$</td>
<td>Probability vector of the desired formation (Definition 2)</td>
</tr>
<tr>
<td>$e_{\text{est}}$</td>
<td>Estimation error between $\mu_k^*$ and $\mu_k^+$ (3)</td>
</tr>
<tr>
<td>$\mu_k^*$</td>
<td>Current swarm distribution (Definition 3)</td>
</tr>
<tr>
<td>$\mu_k^+$</td>
<td>Estimate of $\mu_k^*$ by the $j^{th}$ agent</td>
</tr>
<tr>
<td>$\xi_k^j$</td>
<td>Hellinger-distance-based feedback gain (Definition 6)</td>
</tr>
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C. Notations

The time index is denoted by a right subscript and the agent index is denoted by a lower-case right superscript. Frequently used symbols are listed in Table I.

The symbol $\mathbb{P}(\cdot)$ refers to the probability of an event. Let $\mathbb{N}$ and $\mathbb{R}$ be the set of natural numbers (positive integers) and real numbers respectively. The matrix $\text{diag}(\alpha)$ denotes the diagonal matrix of appropriate size with the vector $\alpha$ as its diagonal elements. Let $1 = [1, 1, \ldots, 1]^T$, $I$, $0$, and $\emptyset$ denote the ones (column) vector, the identity matrix, the zero matrix, and the empty set respectively. Let $\| \cdot \|_p$ denote the $\ell_p$ vector norm. Let $\min^+$ denote the minimum of the positive elements.

A probability vector $\alpha \in \mathbb{R}^n$ is a row vector with non-negative elements, where the sum of all the elements is 1 (i.e., $\alpha \geq 0$, $\alpha 1 = 1$) [46, pp. 92]. The metric $\mathcal{D}(\cdot, \cdot)$ represents the distance between probability vectors $\alpha$ and $\beta$, where the subscript represents the type of metric.

II. PRELIMINARIES AND PROBLEM STATEMENT

In this section, we state the problem statement for shape formation in Section II-B and introduce the PSG–IMC algorithm for shape formation in Section II-C.
element \( r^i_k[i] = 1 \), then the \( j^{th} \) agent is present inside the bin \( B[i] \) at the \( k^{th} \) time instant; otherwise \( r^i_k[i] = 0 \).

**Definition 3.** (Current Swarm Distribution \( \mu^*_k \)) The current swarm distribution \( \mu^*_k \) is a probability vector in \( \mathbb{R}^{n_{bin}} \). It is given by the ensemble mean of actual bin positions of the agents:

\[
\mu^*_k := \frac{1}{m_k} \sum_{j=1}^{m_k} r^j_k. \tag{1}
\]

Each element \( \mu^*_k[i] \) gives the swarm density distribution in the corresponding bin \( B[i] \) at the \( k^{th} \) time instant.

**Definition 4.** (Matrix \( A^k_n \) of Motion Constraints) An agent in a particular bin can only transition to some bins but cannot transition to other bins, because of the dynamics or physical constraints. These (possibly time-varying) motion constraints are specified by the matrix \( A^k_n \in \mathbb{R}^{n_{bin} \times n_{bin}} \), where each element is given by:

\[
A^k_n[i, \ell] = \begin{cases} 1 & \text{if the transition from bin } B[i] \text{ to bin } B[\ell] \text{ is allowed at the } k^{th} \text{ time instant,} \quad (2) \\ 0 & \text{if this transition is not allowed.} \end{cases}
\]

We assume that \( A^k_n \) satisfies the following properties:

1) The matrix \( A^k_n \) is symmetric and irreducible\(^1\),

2) \( A^k_n[i, i] = 1 \) for all agents, bins, and time instants,

3) Since the first \( n_{rec} \) bins are recurrent bins, the sub-matrix \( A^k_n, \text{sub} := A^k_n[1 : n_{rec}, 1 : n_{rec}] \) encapsulate the motion constraints between the recurrent bins. The matrix \( A^k_n, \text{sub} \) is irreducible.

These properties are visualized in Fig. 3.

![Figure 3](image)

(a) Allowed

(b) Not Allowed

Figure 3. In this example, the bins 1 to 4 are recurrent bins. The allowed transitions (motion constraints) are shown as red arrows. Case (a) satisfies all the properties in Definition 4. Case (b) does not satisfy property (3) because the recurrent bins are not strongly connected.

As shown in Fig. 3, the recurrent bins need not be contiguous. Therefore, the desired distribution can have multiple disconnected components. Note that the matrix \( A^k_n \) is different from the Markov matrix introduced in Section III.

**Definition 5.** (Cost Matrix \( C_k \)) Consider a matrix \( C_k \in \mathbb{R}^{n_{bin} \times n_{bin}} \) that encapsulates the cost of transitions between bins. Each element \( C_k[i, \ell] \) denotes the cost incurred by an agent while transitioning from bin \( B[i] \) to bin \( B[\ell] \) at the \( k^{th} \) time instant. This cost represents the control effort or the fuel consumed or any other metric that the agents seek to minimize. We assume that the agents do not incur any cost if they remain in their present bin and the agents incur some positive cost if they transition out of their present bin (i.e., \( C_k[i, i] = 0 \) and \( C_k[i, \ell] > 0 \) for all \( i, \ell \in \{1, \ldots, n_{bin}\} \) and \( i \neq \ell \)).

**Assumption 4.** (A Priori Information Available with Each Agent) We assume that the physical space over which the swarm is distributed, the position of the bins \( B[i] \), and the desired formation shape \( \Theta \) are known before the algorithm starts. Subsequently, the time-varying cost matrices \( C_k \) and the motion constraints matrices \( A^k_n \), which depend on the position of the bins, are computed and stored beforehand. Finally, the four design variables (namely \( \varepsilon_M, \varepsilon_C, \beta_1 \), and \( \tau^1 \)), which are introduced later, are also stored on board each agent.

### A. Distributed Estimation of the Current Swarm Distribution

The algorithms in this paper use feedback of the current swarm distribution \( \mu^*_k \). In order to generate this estimate in a distributed manner, we need the following assumption.

**Assumption 5.** The time-varying communication network topology of the swarm is assumed to be strongly-connected.

Under Assumption 5, several algorithms exist in the literature for estimating \( \mu^*_k \) in a distributed manner [49], [50], [51]. For example, the distributed consensus algorithm [51] is used to estimate \( \mu^*_k \) in Remark 13 in Appendix.

Any distributed estimation algorithm will have some residual estimation error between the current swarm distribution \( \mu^*_k \) and the \( j^{th} \) agent’s estimate of the current swarm distribution at the \( k^{th} \) time instant, which is represented by the probability vector \( \mu^j_k \in \mathbb{R}^{n_{bin}} \). Let the positive parameter \( \epsilon_{\text{est}} \) represent the maximum estimation error between \( \mu^*_k \) and \( \mu^j_k \):

\[
D_{\epsilon_{\text{est}}}(\mu^*_k, \mu^j_k) = \sum_{i=1}^{n_{bin}} |\mu^*_k[i] - \mu^j_k[i]| \leq \epsilon_{\text{est}}, \quad \forall k \in \mathbb{N}. \tag{3}
\]

We later show that our algorithm works remarkably well in the presence of this estimation error \( \epsilon_{\text{est}} \).

### B. Problem Statement for Shape Formation

Under Assumptions 1–5, the objectives of the PSG–IMC algorithm for shape formation are as follows:

- Each agent independently determines its bin-to-bin trajectory using a Markov chain, which obeys motion constraints \( A^k_n \), so that the overall swarm converges to a desired formation shape \( \Theta \).
- The algorithm automatically detects and repairs damages to the formation.
- The algorithm minimizes the expected cost of transitions at every time instant (see Definition 8) for all the agents, where the cost matrix \( C_k \) is defined in Definition 5.

### C. Outline of the PSG–IMC Algorithm

The key steps in the proposed PSG–IMC algorithm for shape formation are shown in Fig. 4. The agent first determines its present bin and estimates the current swarm distribution (Section II–A). If the agent is in a transient bin, then it selects another bin using the condition for escaping transient bins (Section III–C). Otherwise, the agent computes the Markov
The desired formation distance (HD) between the current swarm distribution and the desired transition to bin \( j \) at the \( k \)-th time instant is given by

\[
\xi_{d_{\text{des}}}(k) := \frac{1}{\sqrt{2}} \sum_{i=1}^{n_{\text{bins}}} \left( \sqrt{\Theta[i]} - \sqrt{\mu_{k}[i]} \right)^2.
\]  

The HD is a symmetric measure of the difference between two probability distributions and bounded by 1 [53], [54].

Let \( \xi_{d_{\text{des}}} \) represent the desired convergence error threshold between the final swarm distribution and \( \Theta \).

**Remark 1.** (Advantages of Hellinger Distance) The HD between two probability distributions \( \mu_{k}^{L} \) and \( \Theta \) in (4) is bounded as follows [55]:

\[
\frac{1}{2\sqrt{2}} D_{H}(\Theta, \mu_{k}^{L}) \leq D_{L}(\Theta, \mu_{k}^{L}) \leq \frac{1}{\sqrt{2}} D_{C}(\Theta, \mu_{k}^{L})^{\frac{1}{2}}. \tag{5}
\]

We choose HD, over other popular metrics like \( L_{1} \) and \( L_{2} \) distances, because of its properties illustrated in Fig. 5. The \( L_{1} \) distances for the cases \( (\mu_{1}, \mu_{2}, \mu_{3}) \) from \( \Theta \) are equal. But in Case 1, the wrong agent is in a bin where there should be no agent, hence HD heavily penalizes this case. If all the agents are only in those bins which have positive weights in \( \Theta \), then HD is significantly smaller. Finally, if an agent is missing from a bin that has fewer agents in \( \Theta \) (Case 2) compared to a bin that has more agents in \( \Theta \) (Case 3), then HD penalizes Case 2 slightly more than Case 3. These properties are useful for swarm guidance.

**Figure 4.** Flowchart of the PSG–IMC algorithm for shape formation.

**Figure 5.** In this example, the desired distribution \( \Theta \) has 4 and 2 agents in bins 1 and 2 respectively. In the three cases, one agent (marked in red) is not in its correct bin. The \( L_{1} \) distances are equal, but the HD are different.

Consider the Markov matrix \( M_{k} \) in \( \mathbb{R}^{|n_{\text{bins}}| \times |n_{\text{bins}}|} \) that encapsulates the transition probabilities between bins. Each element \( M_{k}[i, \ell] \) represents the probability that the \( j \)-th agent in bin \( B[i] \) at the \( k \)-th time instant will transition to bin \( B[\ell] \) at the \( (k+1) \)-th time instant:

\[
M_{k}[i, \ell] := \mathbb{P}(r_{k+1}^{i}[\ell] = 1 | r_{k}^{i}[i] = 1). \tag{6}
\]

Therefore, the Markov matrix \( M_{k} \) is row stochastic (i.e., \( M_{k}[i, 1] = 1 \)). Its stationary distribution is defined as follows.

**Definition 7.** (Stationary Distribution) The stationary distribution \( \mathbf{\mu}_{k}^{L} \) of the Markov matrix \( M_{k}^{L} \) is given by the solution of \( \mathbf{\mu}_{k}^{L} = \mathbf{\mu}_{k}^{L} M_{k}^{L} \), where \( \mathbf{\mu}_{k}^{L} \) is a probability (row) vector in \( \mathbb{R}^{|n_{\text{bins}}|} \) (i.e., \( \mathbf{\mu}_{k}^{L} \geq 0, \mathbf{\mu}_{k}^{L}[1] = 1 \)). The stationary distribution is unique if the Markov matrix is irreducible [46, pp. 119].

**Definition 8.** (Expected Cost of Transitions at Each Time Instant) The expected cost of transitions for the \( j \)-th agent at the \( k \)-th time instant is given by \( \sum_{i=1}^{n_{\text{bins}}} \sum_{\ell=1}^{n_{\text{bins}}} C_{j}[i, \ell] M_{k}[i, \ell] \), where the cost matrix \( C_{j} \) is defined in Definition 5.

The following Method 1, Theorem 1, and Corollary 1 present our construction of the optimal Markov matrix \( M_{k} \) that minimizes this expected cost of transitions at each time instant. Our construction technique has no relation with the well-known Metropolis-Hastings (MH) algorithm, which is commonly used for constructing Markov matrices with a given stationary distribution [56], [57]. In the MH algorithm,
the proposal distribution is used to iteratively generate the next sample, which is accepted or rejected based on the desired stationary distribution. There is no direct method for incorporating feedback into the MH algorithm. In contrast, the feedback of the current swarm distribution is directly incorporated within our construction process using the feedback gain.

**Method 1: Construction of Optimal Markov Matrix**

Under Assumptions 1–5, the optimal Markov matrix $M_k^f$ that minimizes the expected cost of transitions at each time instant is constructed as follows:

(A) If $\xi_k^f < \xi_{des}$, then set $M_k^f = I$.

(B) Otherwise, $M_k^f$ is computed as follows:

1. If $A_k^f[i, \ell] = 0$, then set $M_k^f[i, \ell] = 0$ for all bins $i, \ell \in \{1, \ldots, n_{bin}\}$.
2. If $\Theta[\ell] = 0$, then set $M_k^f[i, \ell] = 0$ for all bins $i, \ell \in \{1, \ldots, n_{bin}\}$ with $i \neq \ell$.

The remaining elements in $M_k^f$ are computed using the following linear program (LP):

$$
\text{minimize} \sum_{i=1}^{n_{bin}} \sum_{\ell=1}^{n_{bin}} C_k[i, \ell] M_k^f[i, \ell],
$$

subject to

$$
\begin{align*}
\sum_{\ell=1}^{n_{bin}} M_k^f[i, \ell] &= 1, \quad \forall i, \quad \text{(LP1)} \\
\sum_{i=1}^{n_{bin}} \Theta[i] M_k^f[i, \ell] &= \Theta[\ell], \quad \forall \ell, \quad \text{(LP2)} \\
(1 - \xi_k^f) \leq M_k^f[i, \ell] &\leq 1, \quad \forall i, \quad \text{(LP3)} \\
\varepsilon_M \xi_k^{f} \Theta[\ell] \left(1 - \frac{C_k[i, \ell]}{C_{k,\text{max}} + \varepsilon_C}\right) &\leq M_k^f[i, \ell] \leq \frac{\xi_k^f}{\varepsilon_M}, \quad \forall i \neq \ell, \quad \text{(LP4)}
\end{align*}
$$

where $\varepsilon_M$ is a positive scalar constant in $(0, 1)$, $C_{k,\text{max}}$ is the maximum transition cost (i.e., $C_{k,\text{max}} = \max_{i, \ell} C_k[i, \ell]$), and $\varepsilon_C$ is a positive scalar constant.

**Remark 2.** The Markov matrix $M_k^f$ designed in Method 1 has the following desirable properties:

- If $\xi_k^f < \xi_{des}$ (see Definition 6), the swarm is deemed to have converged to the desired formation. Then, $M_k^f$ is set to an identity matrix so that the agents do not transition anymore. Hence, Step (A) ensures that the swarm remains converged without additional movement.
- If the swarm has not converged to the desired formation (i.e., $\xi_k^f \geq \xi_{des}$), then Step (B) is initiated.
- (CS1) prevents those transitions that are not allowed by motion constraints.
- (CS2) prevents transitions into transient bins.
- The objective function in LP (7) minimizes the expected cost of transitions at the current time instant (see Definition 8).
- (LP1) ensures that $M_k^f$ is row stochastic.
- (LP2) ensures that $M_k^f$ has $\Theta$ as its stationary distribution (i.e., $\Theta M_k^f = \Theta$).
- The lower bound in (LP3) ensures that there is non-zero probability for agents to remain in their present bin when $\xi_k^f < 1$. The upper bound in (LP3) is derived from (LP1).

- The lower bound in (LP4) ensures that the minimum transition probability to a target bin is non-zero and directly proportional to both the feedback gain $\xi_k^f$ and the target bin’s desired distribution $\Theta[\ell]$. But the minimum transition probability decreases with increasing cost of transition to the target bin.
- The upper bound in (LP4) ensures that the maximum transition probability is also directly proportional to the feedback gain $\xi_k^f$.

A salient feature of the constraints (LP3,4) is that they depend on the feedback gain $\xi_k^f$. Therefore, if the swarm distribution $\mu_k^f$ converges to $\Theta$ (i.e., $\mu_k^f \rightarrow \Theta$), then $\xi_k^f \rightarrow 0$ (because $\xi_k^f = D_H(\Theta, \mu_k^f)$ and $M_k^f \rightarrow I$ based on these constraints. The identity matrix ensures that agents settle down after the desired formation is achieved, thereby reducing unnecessary transitions. In Section IV-A, we show that these constraints also help prove the convergence of the algorithm. □

**Theorem 1.** The feasible set of Markov matrices that satisfy the constraints (CS1,2) and the linear constraints in LP (7) in Method 1 is non-empty. The optimal Markov matrix $M_k^f$ is row-stochastic, has $\Theta$ as its stationary distribution, and only allows transitions into recurrent bins.

**Proof:** The optimization problem in (7) is an LP because the constraints are all linear inequalities or equalities and the objective function is linear. An optimal solution for the LP exists if the feasible set of Markov matrices is non-empty. We now show that the following family of Markov matrices $Q_k^f$ are within the set of feasible solutions:

$$
Q_k^f[i, \ell] = \begin{cases} 0 & \text{if } A_k^f[i, \ell] = 0 \\ \frac{\xi_k^f}{\Theta \alpha_k^f} \left(\alpha_k^f[i] \alpha_k^f[\ell] \Theta[\ell]\right) & \text{otherwise} \end{cases} \quad \forall i, \ell \in \{1, \ldots, n_{bin}\}, \quad \text{(8)}
$$

where $\alpha_k^f$ is a positive column vector in $\mathbb{R}^{n_{bin}}$, with $\alpha_k^f \leq \xi_k^f \leq 1$ for all bins. The matrix $Q_k^f$ satisfies (CS1) due to (8). If $\Theta[\ell] = 0$, then the off-diagonal element satisfies $Q_k^f[i, \ell] = 0$ and the matrix $Q_k^f$ satisfies (CS2).

We now show that the matrix $Q_k^f$ satisfies (LP1):

$$
\sum_{i=1}^{n_{bin}} Q_k^f[i, \ell] = \frac{\xi_k^f}{\Theta \alpha_k^f} \sum_{i=1}^{n_{bin}} \alpha_k^f[i] \Theta[\ell] + 1 - \xi_k^f \alpha_k^f[i] = 1,
$$

where $\sum_{i=1}^{n_{bin}} \left(\alpha_k^f[i] \Theta[\ell]\right) = \Theta \alpha_k^f$. We now prove that the matrix $Q_k^f$ satisfies (LP2):

$$
\sum_{i=1}^{n_{bin}} \Theta[i] Q_k^f[i, \ell] = \frac{\xi_k^f}{\Theta \alpha_k^f} \sum_{i=1}^{n_{bin}} \left(\alpha_k^f[i] \Theta[\ell]\right) + \left(\Theta[\ell] - \xi_k^f \Theta[\ell] \alpha_k^f[\ell]\right) = \Theta[\ell].
$$
The matrix $Q_k^j$ satisfies (LP3) because each diagonal element $Q_k^j[i, i]$ is lower bounded by $\left(1 - \xi_k^j \alpha_k^j[i]\right)$, which is one of the positive terms in (9). The term $\left(1 - \xi_k^j \alpha_k^j[i]\right)$ is lower bounded by $(1 - \xi_k^j)$ because $\alpha_k^j[i] \leq 1$.

We now prove that the matrix $Q_k^j$ satisfies (LP4). Since the element $\Theta[\ell] > 0$, the off-diagonal element $Q_k^j[i, \ell]$ is upper bounded by $\frac{\xi_k^j}{\epsilon_M}$ because $\alpha_k^j[i] \leq 1$, $\Theta[\ell] \leq 1$, and $(\frac{\xi_k^j}{\epsilon_M}) \leq \frac{\xi_k^j}{\epsilon_M} \leq \frac{\xi_k^j}{\epsilon_M}$. On the other hand, the off-diagonal element $Q_k^j[i, \ell]$ is lower bounded by $\epsilon_M \xi_k^j \Theta[\ell]$ because $\alpha_k^j[i] \geq \epsilon$ and $(\frac{\xi_k^j}{\epsilon_M}) \geq \xi_k^j$ as $\Theta[\ell] = 1$.

Therefore, $\epsilon_M \xi_k^j \Theta[\ell] \left(1 - \frac{C_k^j[i, \ell]}{C_{k, max} + C} \right) \leq Q_k^j[i, \ell]$ because $\left(1 - \frac{C_k^j[i, \ell]}{C_{k, max} + C} \right) < 1$. Thus the matrix $Q_k^j$ satisfies (LP4).

Therefore, the feasible set is non-empty and the optimal Markov matrix $M_k^j$ possesses the desirable properties discussed in Remark 2.

---

**Figure 6.** Multiresolution images of the Eiffel Tower are shown, where the spatial resolution increases from (a) to (d). All the bins are shown in (a) and (b), whereas only a few bins are shown in the left-top corner in (c) and (d).

**Remark 3. (Computation Time)** Although each agent only needs the row of the Markov matrix $M_k^j$ corresponding to its present bin, it has to solve the entire LP (7). The computation time for an LP increases with an increasing number of bins because the number of variables in $M_k^j$ is approximately equal to $n_{bin}^2$. For example, if the desired formation is given by $\nu_{25}$ or $\nu_{75}$ in Fig. 6, then the computation time is a few minutes on a standard desktop computer. If the desired formation is given by $\nu_{150}$ (with $5 \times 10^8$ variables) or $\nu_{300}$ (with $8 \times 10^8$ variables), then the LP is impractical for real-time computation. This escalating computation time with increasing number of bins is an issue for all LP-based algorithms [27, 36].

Therefore, we need a faster method for computing the Markov matrices. The following corollary gives the analytical formula of the optimal Markov matrix when the cost matrix is symmetric.

**Corollary 1.** The optimal Markov matrix of the LP (7) in Method 1 is given by:

\[
M_k^j[i, \ell] = \begin{cases} 
0 & \text{if } A_k^j[i, \ell] = 0 \\
\epsilon_M \xi_k^j \Theta[\ell] \left(1 - \frac{C_k^j[i, \ell]}{C_{k, max} + C} \right) & \text{otherwise,} \\
\end{cases}
\]

\forall i, \ell \in \{1, \ldots, n_{bin}\} \text{ and } i \neq \ell, \quad (10)

\[
M_k^j[i, i] = 1 - \sum_{\ell \in \{1, \ldots, n_{bin}\} \setminus \{i\}} M_k^j[i, \ell]. \quad (11)
\]

if the cost matrix $C_k$ is symmetric (i.e., $C_k = C_k^T$).

---

**Proof:** We first state a simpler LP by neglecting the constraints (LP1,2) from the original LP (7). We state this simpler LP (12) using the following substitutions for all positive elements $R_k^j[i, \ell] = M_k^j[i, \ell] - (1 - \xi_k^j)$ and $R_k^j[i, \ell] = M_k^j[i, \ell] - \epsilon_M \xi_k^j \Theta[\ell] \left(1 - \frac{C_k^j[i, \ell]}{C_{k, max} + C} \right)$:

\[
\begin{aligned}
&\text{minimize} \sum_{i=1}^{n_{bin}} \sum_{\ell=1}^{n_{bin}} C_k[i, \ell] R_k^j[i, \ell] \\
&\text{subject to:} \\
&0 \leq R_k^j[i, \ell] \leq \xi_k^j, \forall i, \quad (\widetilde{LP}3) \\
&0 \leq R_k^j[i, \ell] \leq \epsilon_M \xi_k^j \Theta[\ell] \left(1 - \frac{C_k^j[i, \ell]}{C_{k, max} + C} \right), \quad (LP4)
\end{aligned}
\]

According to Definition 5, $C_k[i, \ell] = 0$ and $C_k[i, \ell] > 0$ for all $i \neq \ell$. The minimum cost of this simpler LP (12) is obtained when $\sum_{i=1}^{n_{bin}} \sum_{\ell=1}^{n_{bin}} C_k[i, \ell] R_k^j[i, \ell] = 0$, because the second term in the cost function is a constant. Therefore, all the positive off-diagonal elements $M_k^j[i, \ell]$ are equal to their respective lower bounds $\epsilon_M \xi_k^j \Theta[\ell] \left(1 - \frac{C_k^j[i, \ell]}{C_{k, max} + C} \right)$ in the optimal solution of the simpler LP (12). This optimal solution of the simpler LP (12) is given by the Markov matrix $M_k^j$ (10)-(11).

If the optimal solution of the simpler LP (12) also satisfies the constraints (LP1,2) that we neglected previously, then it is the optimal solution of the original LP (7). It follows from the construction of the diagonal elements in $M_k^j$ (11) that it satisfies (LP1). The diagonal elements of $M_k^j$ are given by:

\[
M_k^j[i, i] = 1 - \sum_{\ell \in \{A_k^j[i, \ell] = 1, \ell \neq i\}} \epsilon_M \xi_k^j \Theta[\ell] \left(1 - \frac{C_k^j[i, \ell]}{C_{k, max} + C} \right). \quad (13)
\]

Note that the matrix $M_k^j$ is a reversible Markov matrix because of the symmetric cost matrix, i.e., $\Theta[\ell] M_k^j[i, \ell] = \Theta[\ell] M_k^j[i, \ell] = \epsilon_M \xi_k^j \Theta[\ell] \left(1 - \frac{C_k^j[i, \ell]}{C_{k, max} + C} \right)$ for all $i, \ell$.

Hence, the matrix $M_k^j$ also satisfies (LP2) because:

\[
\sum_{i=1}^{n_{bin}} \Theta[\ell] M_k^j[i, \ell] = \sum_{i=1}^{n_{bin}} \Theta[\ell] M_k^j[i, \ell] = \Theta[\ell] \left(\sum_{i=1}^{n_{bin}} M_k^j[i, \ell]\right) = \Theta[\ell].
\]

Therefore, the matrix $M_k^j$ is the optimal solution of the original LP (7).

---

If the cost matrix $C_k$ is symmetric, then the Markov matrix $M_k^j$ (10)-(11) gives significant savings in computation time because each agent can directly compute the row of the optimal Markov matrix. For example, the computation times for all four cases in Fig. 6 are less than 2 minutes on a standard desktop computer.

**Remark 4. (Alternative Functions for Constraints)** Note that our construction technique holds even if the term $\left(1 - \frac{C_k^j[i, \ell]}{C_{k, max} + C} \right)$ in the constraint (LP4) in Method 1 and
(10) in Corollary 1 is replaced by any monotonic function in \((0, 1]\) that decreases with an increasing \(C_k[i, \ell]\). Similarly, the term \(\xi_k^\ell\) in the constraints (LP3,4) can be replaced by any monotonic function in \((0, 1]\) that decreases with a decreasing \(\xi_k^\ell\). For example, see Fig. 12(b) in Section VI-B.

\[ \square \]

B. Construction of Fastest Mixing IMC

In this subsection, we construct the fastest mixing IMC, where the IMC’s convergence rate to the rank one matrix \(1\Theta\) is optimized. The convergence rate of HMC, with time-invariant Markov matrix \(M\), is determined by the second largest eigenvalue modulus (i.e., \(\max_{\zeta \in \{2, \ldots, n_{\text{rec}}\}} |\lambda_2(M)|\)) [58], [59]. On the other hand, the convergence rate of IMC is determined by the coefficient of ergodicity [46, pp. 137]. Since the first \(n_{\text{rec}}\) bins are recurrent bins, the Markov matrix \(M_k\) can be decomposed as follows:

\[
M_k = \begin{bmatrix}
M_{\text{sub}}^k & Q^{n_{\text{rec}} \times (n_{\text{max}} - n_{\text{rec}})} \\
M_{\text{rec}}^k & M_{\text{sub}}^k
\end{bmatrix},
\]

where the sub-matrix \(M_{\text{sub}}^k := M_{\text{sub}}^k[1 ; n_{\text{rec}}, 1 ; n_{\text{rec}}]\) encapsulates the bin transition probabilities between the recurrent bins.

**Definition 9.** [46, pp. 137–139] (Coefficient of Ergodicity)

For the stochastic matrix \(M_{\text{sub}}^k\), the coefficient of ergodicity \(\tau_1(M_{\text{sub}}^k)\) is defined as:

\[
\tau_1(M_{\text{sub}}^k) = \sup_{v_1, v_2, v_1 \neq v_2} \frac{D_{c_1}(v_1 M_{\text{sub}}^k, v_2 M_{\text{sub}}^k)}{D_{c_1}(v_1, v_2)},
\]

\[
= 1 - \min_{i, \ell} \sum_{s=1}^{n_{\text{rec}}} \min \left( M_{\text{sub}}^k[i, s], M_{\text{sub}}^k[\ell, s] \right),
\]

where \(v_1, v_2\) are probability row vectors in \(\mathbb{R}^{n_{\text{rec}}}\) and \(i, \ell, s \in \{1, \ldots, n_{\text{rec}}\}\).

We define \(n_{k, \text{dia}}^j\) as the graph diameter in the graph conforming to the matrix \(A_{k, \text{sub}} := A_{k, \text{sub}}[1 ; n_{\text{rec}}, 1 ; n_{\text{rec}}]\); i.e., it is the greatest number of edges in a shortest path between any pair of recurrent bins [60]. If \(n_{k, \text{dia}}^j > 2\), then there exist recurrent bins \(B[i]\) and \(B[\ell]\) such that either \(M_{\text{sub}}^k[i, s] = 0\) or \(M_{\text{sub}}^k[\ell, s] = 0\) for all \(s \in \{1, \ldots, n_{\text{rec}}\}\). Substituting these bins into (15) show that \(\tau_1(M_{\text{sub}}^k) = 1\) when \(n_{k, \text{dia}}^j > 2\). In order to avoid this trivial case, we minimize the coefficient of ergodicity of the positive matrix \((M_{\text{sub}}^k)^{n_{k, \text{dia}}^j}\) [61, Theorem 8.5.2, pp. 516].

**Corollary 2.** (Construction of Fastest-Mixing Markov Matrix)

The following convex optimization problem is used instead of the LP (7) in Method 1:

\[
\min \tau_1 \left( (M_{\text{sub}}^k)^{n_{k, \text{dia}}^j} \right),
\]

subject to (LP1 – 4) in (7), where \(\tau_1\) is defined in Definition 9.

**Proof:** The cost function \(\tau_1 \left( (M_{\text{sub}}^k)^{n_{k, \text{dia}}^j} \right)\) is a convex function of the stochastic matrix \(M_{\text{sub}}^k\), because it can be expressed as [46, Lemma 4.3, pp. 139]:

\[
\tau_1 \left( (M_{\text{sub}}^k)^{n_{k, \text{dia}}^j} \right) = \sup_{\|\delta\|_2 = 1, \delta_1 = 0} \left\| \delta \cdot (M_{\text{sub}}^k)^{n_{k, \text{dia}}^j} \right\|_1,
\]

where \(\delta = \text{const}(v_1 - v_2)\) is a row vector in \(\mathbb{R}^{n_{\text{rec}}}\). Hence, (16) is a convex optimization problem and the family of Markov matrices \(Q_k^{(8)-(9)}\) is a subset of its feasible set.

\[ \square \]

C. Condition for Escaping Transient Bins

In this subsection, we discuss the condition for escaping transient bins.

**Definition 10.** (Trapping Bins) If an agent is inside a transient bin \(\Theta[i] = 0\) and its motion constraints matrix \(A_k^i\) only allows transitions to other transient bins, then that transient bin is called a trapping bin. This agent is trapped in this bin because the Markov matrix does not allow transitions out of this bin. Let \(T_k\) represent the set of trapping bins for the \(j^\text{th}\) agent at the \(k^\text{th}\) time instant. For example, see Fig. 7.

\[ \square \]

Figure 7. In this example, the bins 1 and 2 are recurrent bins. The allowed transitions (motion constraints) are shown in red. The trapping bins for the two cases are enclosed in blue.

Since the irreducible motion constraints matrices \(A_k^i\) are known a priori (Assumption 4 and Definition 4), the deterministic path for exiting the set of trapping bins is stored on board each agent. For each trapping bin \(B[i] \in T_k\), the \(j^\text{th}\) agent transitions to a transient bin \(\Psi_k^j[i]\), chosen a priori, such that the transition from bin \(B[i]\) to bin \(\Psi_k^j[i]\) is allowed by motion constraints. This deterministic path, which is chosen a priori, ensures that the agent exits the set of trapping bins as soon as possible. This bin \(\Psi_k^j[i]\) has to be chosen on a case-by-case basis depending on the motion constraints matrix \(A_k^i\). For example, in Fig. 7, for the trapping bin 5, the best option is bin 3 in case (a) and bin 7 in case (b). Therefore, the agent can follow this path to deterministically exit the set of trapping bins in finite time instants.

If an agent is in a transient bin, but not in a trapping bin, then its motion constraint matrix allows transitions to some recurrent bins. We can speed up the process of exiting this transient bin by forcing the agent to transition to any reachable recurrent bin, with equal probability, during the current time instant. Thus the agent transitions from its current transient bin to a recurrent bin in one time instant.

The matrix \(S_k^j \in \mathbb{R}^{n_{\text{rec}} \times n_{\text{bin}}}\) encapsulates the condition for escaping transient bins. If \(B[i]\) is a transient bin (i.e., \(\Theta[i] = 0\)), then each element in the corresponding row \(S_k^j[i, 1 : n_{\text{bin}}]\) is given by:

\[
S_k^j[i, \ell] = \begin{cases} 1 & \text{if } B[i] \in T_k^j \text{ and } B[\ell] = \Psi_k^j[i] \\ \frac{1}{n_{k,i}^j} & \text{if } B[i] \not\in T_k^j \text{ and } A_k^i[i, \ell] = 1 \text{ and } \Theta[\ell] > 0 \\ 0 & \text{otherwise} \end{cases},
\]

where \(n_{k,i}^j\) is the number of recurrent bins that the \(j^\text{th}\) agent can transition to, from bin \(B[i]\) at the \(k^\text{th}\) time instant. This
IV. PSG–IMC ALGORITHM FOR SHAPE FORMATION

In this section, we first state the PSG–IMC algorithm for shape formation and then present its convergence analysis and its property of robustness.

Method 2: PSG–IMC Algorithm for Shape Formation

1. One iteration of the $i$th agent during $k$th time instant, where the $j$th agent is in bin $B[i]$
2. Given $\Theta$, $C_k$, $A_k^i$, $\mu_k^i$, $\epsilon_M$, $\epsilon_C$, $\tau$, and $\beta$
3. if $\Theta[i]=0$, then
4. Compute $S_k'[i, 1:n_{bin}]$ using (17)
5. Generate a random number $z \in \text{unif}[0; 1]$
6. Select bin $B[q]$ such that $\sum_{\ell=1}^{q-1} S_k'[i, \ell] \leq z < \sum_{\ell=1}^{q} S_k'[i, \ell]$
7. else
8. Compute the feedback gain $\xi_k^i$ using (4)
9. Compute $M_k^i[i, 1:n_{bin}]$ using Corollary 1 or compute $M_k^i$ using Method 1
10. Compute the term $\eta_{k,i}^j$ using (20)
11. Compute $P_k^j[i, 1:n_{bin}]$ using (18) and (19)
12. Generate a random number $z \in \text{unif}[0; 1]$
13. Select bin $B[q]$ such that $\sum_{\ell=1}^{q-1} P_k^j[i, \ell] \leq z < \sum_{\ell=1}^{q} P_k^j[i, \ell]$
14. end if
15. Go to bin $B[q]$ while avoiding collisions

The pseudo-code for the PSG–IMC algorithm for shape formation is given in Method 2, whose key steps are shown in Fig. 4. At the start, the $i$th agent knows the desired formation shape $\Theta$, the time-varying cost matrix $C_k$, and its time-varying motion constraints matrix $A_k^i$ (Assumption 4). During each iteration, the agent determines the bin it belongs to (Assumption 3, here we assume that the $j$th agent is in bin $B[i]$) and the current swarm distribution $\mu_k^i$ from Section II-A (lines 1–2).

If the agent is in a transient bin (line 3), then it uses inverse transform sampling (Remark 14 in Appendix) to select the next bin $B[q]$ from the corresponding row of the matrix $S_k'[i, 1:n_{bin}]$ (lines 4–6).

Otherwise, the agent first computes the HD-based feedback gain $\xi_k^i$ (line 8). If the cost matrix $C_k$ is symmetric, the agent can directly compute the row $M_k^i[i, 1:n_{bin}]$ using Corollary 1 (line 9). Otherwise, the agent computes the entire Markov matrix $M_k^j$ using Method 1 (line 9).

In order to avoid undesirable transitions from bins that are deficient in agents (i.e., where $\Theta[i] > \mu_k^i[i]$), the agent modifies its Markov matrix row $M_k^j[i, 1:n_{bin}]$ as follows:

$$P_k^j[i, \ell] = \begin{cases} 1 - \eta_{k,i}^j & \forall i \neq \ell \\ M_k^j[i, \ell] & \end{cases}$$

and

$$P_k^j[i, i] = \begin{cases} 1 - \eta_{k,i}^j & M_k^j[i, i] + \eta_{k,i}^j \\ \end{cases}$$

where $\eta_{k,i}^j = \exp(-\tau_k)$, $\exp(\beta_i (\Theta[i] - \mu_k^i[i]))$ and $\tau_k$ and $\beta_i$ are time-invariant positive constants (lines 10–11). Then, the agent uses inverse transform sampling (Remark 14 in Appendix) to select the next bin $B[q]$ from the bin transition probabilities $P_k^j[i, 1:n_{bin}]$ (lines 12–13). Finally, the agent goes to the selected bin $B[q]$ from the current bin $B[i]$ in a collision-free manner using the lower-level guidance and control algorithm discussed in Remark 15 in Appendix (line 15).

Remark 5. (Explanation of the term $\eta_{k,i}^j$) The term $\eta_{k,i}^j$ (20) is designed to suppress the transition probabilities from a bin that is deficient in agents; i.e., if $\Theta[i] > \mu_k^i[i]$, then $\eta_{k,i}^j = \exp(-\tau_k)$. Its effect decreases with increasing time instants. The design variables $\beta_i$ and $\tau_k$ dictate the amplitude and time constant of this suppression. It is shown later in Section VI-A that the undesirable transitions, suppressed using this term, greatly reduce the total number of transitions, which in turn significantly improves the convergence error.

Remark 6. (PSG–IMC Algorithm is a Distributed Algorithm) Under Assumptions 3 and 5, the agent perceives the bin it belongs to and estimates the current swarm distribution in a distributed manner. The remaining terms in lines 1–2 are known a priori. Lines 3–14 are executed individually by each agent. Finally, in line 15, the agent only needs to communicate with its neighboring agents as shown in Remark 15 in Appendix. Thus, the steps in Method 2 can be accomplished in a distributed manner.

A. Main Result: Convergence Analysis

In this subsection, we prove that the swarm distribution $\mu_k^*_{\text{des}}$ converges to the desired formation shape $\Theta$ with prescribed convergence errors using the PSG–IMC algorithm for shape formation given in Method 2. Unlike the convergence proof for HMC, which is a direct application of the Perron–Frobenius theorem, the convergence proof for IMC is rather involved (e.g., see [46], [62]). We first state an assumption on $\xi_k^i$ for this subsection.

Assumption 6. (Minimum Value of $\xi_k^i$) If $\xi_k^i < \xi_{\text{des}}$, then the current swarm distribution is sufficiently close to the desired formation (see Definition 6). Moreover, the Markov matrices in Method 1 become an identity matrix, hence the agents do not transition any more. Therefore, the swarm has converged to the desired formation and no further convergence is necessary.

Therefore, in this subsection, we assume that the swarm has not converged to the desired formation (i.e., $\xi_k^i \geq \xi_{\text{des}}$).

We first show that agents in recurrent bins transition using the modified Markov matrix $P_k^j$ derived from (18)-(19).
Theorem 2. According to Method 2, if an agent is in a recurrent bin, then it transitions using the following modified Markov matrix $P^j_k$ from (18)-(19):

$$P^j_k = (1 - D^j_k) M^j_k + D^j_k,$$

where $D^j_k = \text{diag} \left( \nu^j_{k,1}, \ldots, \nu^j_{k,n_{\text{bin}}} \right)$. The Markov matrix $P^j_k$ is row stochastic (i.e., $P^j_k 1 = 1$), asymptotically homogeneous with respect to $\Theta$ (i.e., $\lim_{k \to \infty} P^j_k = \Theta$), and only allows transitions into recurrent bins.

Proof: The modified Markov matrix $P^j_k$ (21) is derived from (18)-(20). It follows from lines 3 and 6 of Method 2 that the agent uses the Markov matrix $P^j_k$ to transition if and only if it is in a recurrent bin (i.e., $\Theta[i] > 0$).

The matrix $P^j_k$ is row stochastic because $M^j_k 1 = 1$. The matrix $M^j_k$ has $\Theta$ as its stationary distribution for all $k \in \mathbb{N}$. It follows from the definition of the term $\nu^j_{k,i}$ (20) that $\lim_{k \to \infty} D^j_k = 0$ because $\lim_{k \to \infty} \exp(-\tau^j_k) = 0$. Therefore, $\lim_{k \to \infty} P^j_k = \lim_{k \to \infty} M^j_k$. Hence, the sequence of matrices $P^j_k$ is asymptotically homogeneous with respect to $\Theta$ because $\lim_{k \to \infty} \Theta P^j_k = \lim_{k \to \infty} \Theta M^j_k = \Theta$ (see Definition 13 in Appendix).

Note that the element $P^j_k[i, \ell] > 0$ if and only if the corresponding element $M^j_k[i, \ell] > 0$ for all $i, \ell \in \{1, \ldots, n_{\text{bin}}\}$ and $k \in \mathbb{N}$. Therefore, like the matrix $M^j_k$, the matrix $P^j_k$ only allows transitions into recurrent bins.

We now show that all the agents leave the transient bins and enter the recurrent bins in finite time instants.

Theorem 3. According to Method 2, each agent is in a recurrent bin by the $T^{th}$ time instant, where $T \leq (n_{\text{bin}} - n_{\text{rec}}) + 1$. Once an agent is inside a recurrent bin, it always remains within the set of recurrent bins.

Proof: If an agent is in a recurrent bin, then it follows from Theorem 2 that it cannot transition to any transient bin.

If the agent is in a trapping bin, then the matrix $S^j_k$ (17) ensures that the agent exits the set of trapping bins as soon as possible in a deterministic manner. Therefore, the maximum number of steps inside the set of trapping bins is upper bounded by the number of transient bins $(n_{\text{bin}} - n_{\text{rec}})$.

If an agent is in a transient bin, but not in a trapping bin, then the matrix $S^j_k$ (17) ensures that the agent transitions to a recurrent bin in one time instant. Hence each agent enters a recurrent bin in at most $(n_{\text{bin}} - n_{\text{rec}}) + 1$ time instants.

Consider a probability (row) vector $x^j_k \in \mathbb{R}^{n_{\text{bin}}}$, which denotes the probability mass function (PMF) of the predicted bin position of the $j^{th}$ agent at the $k^{th}$ time instant. Each element $x^j_k[i]$ gives the probability that the $j^{th}$ agent is in bin $B[i]$ at the $k^{th}$ time instant:

$$x^j_k[i] = P(r^j_k[i] = 1), \quad \forall i \in \{1, \ldots, n_{\text{bin}}\}.$$

We now discuss convergence of each agent’s predicted bin position $x^j_k$ to the desired formation $\Theta$.

Theorem 4. (Convergence of IMC) According to Method 2, each agent’s time evolution of the vector $x^j_k$ converges pointwise to the desired stationary distribution $\Theta$ irrespective of the initial condition, i.e., $\lim_{k \to \infty} x^j_k = \Theta$ pointwise for all agents.

Proof: It follows from Theorem 3 that all agents are always in the set of recurrent bins from the $T^{th}$ time instant onwards. Since the first $n_{\text{rec}}$ bins are recurrent bins, we decompose the vector $x^j_k = [x^j_k, 0, \ldots, 0]$ for all $k \geq T$, where the probability row vector $x^j_k := [x^j_k[1], \ldots, x^j_k[n_{\text{rec}}]] \in \mathbb{R}^{n_{\text{rec}}}$ captures the agent’s PMF over the set of recurrent bins. Similarly, we decompose $\Theta = [\Theta, 0, \ldots, 0]$, where $\Theta := [\Theta[1], \ldots, \Theta[n_{\text{rec}}]]$. Note that convergence of $x^j_k$ to $\Theta$, implies the convergences of $x^j_k$ to $\Theta$.

According to Theorem 2, the time evolution of the PMF vector $x^j_k$ is given by:

$$x^j_{k+1} = x^j_k P^j_k \Theta, \quad \forall k \geq T,$$

where the row-stochastic sub-matrix $P^j_k \Theta := P^j_k[1 : n_{\text{rec}}, 1 : n_{\text{rec}}]$ encapsulates the bin transition probabilities between the recurrent bins. The matrix $P^j_k \Theta$, like the matrix $M^j_k$ in (14), is irreducible because the matrix $A^j_{k,\text{sub}}$ is irreducible (Definition 4).

It follows from (4) that $D_H(\Theta, \mu^j_k) = 1$ if and only if $\mu^j_k[i] = 0$ for all recurrent bins $i \in \{1, \ldots, n_{\text{rec}}\}$, because $\Theta[i] > 0$ only in recurrent bins (Definition 2). Therefore, the feedback gain $\xi^j_k < 1$ for all time instant $k \geq T$ because all agents are in recurrent bins. Hence, the diagonal elements $M^j_{k,\text{sub}}[i, i]$ and $P^j_{k,\text{sub}}[i, i]$ are positive for all $i \in \{1, \ldots, n_{\text{rec}}\}$ and $k \geq T$.

The overall time evolution of the agent’s PMF vector is given by the IMM for all $r > T$:

$$x^j_r = x^j_T P^j_{T+1,T} \cdots P^j_{r-1,T} = x^j_T U^j_{T,r}.$$

We now show that this forward matrix product $U^j_{T,r}$ is strongly ergodic (see Definition 14 in Appendix) and $\Theta$ is its unique limit vector (i.e., $\lim_{r \to \infty} U^j_{T,r} = \Theta$).

The matrix $U^j_{T,r}$ is a product of nonnegative matrices, hence it is a nonnegative matrix. If $P^j_{k,\text{sub}}[i, \ell] > 0$ for some $k \in \{T, \ldots, r-1\}$ and $i, \ell \in \{1, \ldots, n_{\text{rec}}\}$, then the corresponding element $U^j_{T,r}[i, \ell] > 0$ because, as shown below, the value of $U^j_{T,r}[i, \ell]$ is lower bounded by the product of positive diagonal elements and $P^j_{k,\text{sub}}[i, \ell]$:

$$U^j_{T,r}[i, \ell] \geq P^j_{T,T}[i, \ell] \left( \prod_{q=T+1}^{r-1} P^j_{q,\text{sub}}[i, \ell] \right) + \sum_{s=T+1}^{r-2} \left( \prod_{q=T}^{s-1} P^j_{q,\text{sub}}[i, \ell] \right) P^j_{s,\text{sub}}[i, \ell] \left( \prod_{q=s+1}^{r-1} P^j_{q,\text{sub}}[i, \ell] \right), \quad \text{if } i \neq \ell,$$

$$U^j_{T,r}[i, i] \geq \left( \prod_{q=T}^{r-1} P^j_{q,\text{sub}}[i, i] \right), \quad \text{if } i = \ell.$$

Therefore, the matrix $U^j_{T,r}$, like the matrix $P^j_{k,\text{sub}}$, is irreducible because $U^j_{T,r}[i, \ell] > 0$ if $P^j_{k,\text{sub}}[i, \ell] > 0$ for all $i, \ell \in \{1, \ldots, n_{\text{rec}}\}$. Since the irreducible matrix $U^j_{T,r}$ has
positive diagonal elements (26), it is a primitive matrix \cite[Lemma 8.5.4, pp. 516]{61}.

Some of off-diagonal elements in $M_{\text{k, sub}}^j$ and $P_{\text{k, sub}}^j$ are zero due to the constraints (CS1,2) in Theorem 1. The lower bound $\gamma^j$, which is independent of $k$, for the remaining positive elements in $P_{\text{k, sub}}^j$ is given by the constraint (LP4) in Theorem 1, the lower bound of $\xi^j$ in Assumption 6, and the upper bound of the term $\eta^j_{k, i}$ (20):

$$
\gamma^j = (1 - \exp(-\tau^j T)) \xi_{\text{des}} \in M \left( \min_{i} \Theta \right) \left( 1 - \frac{C_{\text{max}}}{\xi_{\text{max}} + \varepsilon_C} \right)
\leq \min_{i, \ell} \max_{k, \ell} P_{\text{k, sub}}^j[i, \ell], \quad k \geq T ,
$$

(27)

where $\min_{\tau}$ refers to the minimum of the positive elements and $C_{\text{max}} = \max_{k \in \mathbb{N}} C_{k, \text{max}}$. It follows from Theorem 2 that the sequence of matrices $P_{\text{k, sub}}^j$, $k \geq T$ is asymptotically homogeneous with respect to $\Theta$. Since (i) the forward matrix product $U_{T, r}$ is primitive and (ii) there exists $\gamma^j$ (independent of $k$), it follows from Theorem 8 in Appendix that the forward matrix product $U_{T, r}$ is strongly ergodic. Since (i) the matrices $P_{\text{k, sub}}^j$, $k \geq T$ are irreducible and (ii) there exists $\gamma^j$ (independent of $k$), it follows from Theorem 9 in Appendix that the limit vector $e = \Theta$. Since $U_{T, r}$ is strongly ergodic, it follows from Corollary 3 in Appendix that the unique limit vector is given by $\Theta$ (i.e., $\lim_{r \to \infty} U_{T, r}^j = 1 \Theta$). Hence, each agent’s PMF vector converges to:

$$
\lim_{r \to \infty} \vec{x}_{T}^j \approx \lim_{r \to \infty} \vec{x}_{T}^j U_{T, r}^j = \vec{x}_{T}^j 1 \Theta = \Theta .
$$

Therefore, $\lim_{k \to \infty} \vec{x}_{T}^j = \Theta$ pointwise for all agents.

**Theorem 5.** Since $\lim_{r \to \infty} U_{T, r}^j = 1 \Theta$, for all $\varepsilon \in \mathbb{N}$, there exists a $k_{\text{lim}} \in \mathbb{N}$ such that $D_{L_1}(\Theta U_{T, r}^j, \Theta) \leq \varepsilon \in \mathbb{N}$ for all $r \geq k_{\text{lim}}$. The convergence error between the $j$th agent’s PMF vector $\vec{x}_{T}^j$ and the desired formation $\Theta$ is bounded for all $r \geq k_{\text{lim}}$ by:

$$
D_{L_1}(\vec{x}_{T}^j, \Theta) \leq \varepsilon \in \mathbb{N} \left[ \left( \frac{\tau^j T}{\tau^j T + s(n_{\text{rec}} - 1)} \right)^{-1} \left( \prod_{q=T+1}^{n_{\text{rec}} - 1} \delta_q^j \right)^{\min \left\{ i, \ell + P_{i, \text{sub}}^j[i, \ell] \right\} } \right] .
$$

where $[\cdot]$ is the floor function and $\delta_q^j = \min \left\{ i, \ell + P_{j, \text{sub}}^j[i, \ell] \right\}$. \hspace{1cm} □

**Proof:** It follows from the definition of $\tau_1(U_{T, r}^j)$ in (15) that:

$$
D_{L_1}(\vec{x}_{T}^j U_{T, r}^j, \Theta) \leq \tau_1(U_{T, r}^j) D_{L_1}(\vec{x}_{T}^j, \Theta). \hspace{1cm} (28)
$$

Since $\vec{x}_{T}^j = \vec{x}_T^j U_{T, r}^j$, (24), we get the triangle inequality:

$$
D_{L_1}(\vec{x}_{T}^j, \Theta) \leq D_{L_1}(\vec{x}_{T}^j U_{T, r}^j, \Theta) + D_{L_1}(\Theta U_{T, r}^j, \Theta) \leq \tau_1(U_{T, r}^j) D_{L_1}(\vec{x}_{T}^j, \Theta) + \varepsilon \in \mathbb{N}.
$$

The sub-multiplicative property of $\tau_1(U_{T, r}^j)$ \cite[Lemma 4.3, pp. 139]{61} gives:

$$
\tau_1(U_{T, r}^j) \leq \prod_{s=0}^{\frac{\tau^j T}{\tau^j T + s(n_{\text{rec}} - 1)} - 1} \tau_1(U_{T+1}^{T+s(n_{\text{rec}} - 1), T+s(n_{\text{rec}} - 1)}) .
$$

Here, if $r > T + \frac{s-T}{n_{\text{rec}} - 1}$, then we neglect the contribution of the residual term by assuming $\tau_1(U_{T+1}^{T+s(n_{\text{rec}} - 1), T+s(n_{\text{rec}} - 1)}) = 1$.

The matrix $U_{T + s(n_{\text{rec}} - 1), T+s(n_{\text{rec}} - 1)}$, for any $k \geq T$, is a positive matrix because there exists a path of length smaller than $(n_{\text{rec}} - 1)$ between every two recurrent bins (see Theorem 10 in Appendix). A conservative lower bound on the elements in the positive matrix $U_{T + s(n_{\text{rec}} - 1), T+s(n_{\text{rec}} - 1)}$ is given by the product of the smallest positive elements in each of the matrices, i.e.,

$$
U_{T+s(n_{\text{rec}} - 1), T+s(n_{\text{rec}} - 1)}^{i, j} \geq \left( \prod_{q=T+1}^{n_{\text{rec}} - 1} \delta_q^j \right) \max_{k \in \mathbb{N}} C_{k, \text{max}}.
$$

(29)

We now focus on the convergence of the swarm distribution to the desired formation. In practical scenarios, the number of agents is finite, hence the following theorem gives a lower bound on the number of agents.

**Theorem 6.** Let $\varepsilon \in \mathbb{N}$, $\varepsilon \in \mathbb{N}$, and $\varepsilon \in \mathbb{N}$ represent convergence error thresholds. Let $\kappa$ represent the latest time instant when an agent is added to or removed from the swarm, i.e., the number of agents $m_k = m_k$ for all $k \geq \kappa$. Since $\lim_{k \to \infty} U_{T, r}^{j} = 1 \Theta$ for all agents, there exists $k_{\text{lim}} \in \mathbb{N}$ such that $D_{L_1}(\Theta U_{T, r}^{j} k_{\text{lim}}, \Theta) \leq \varepsilon \in \mathbb{N}$ for all $k \geq k_{\text{lim}}$ and $j \in \{1, \ldots, m_k\}$. The convergence error between the swarm distribution $\mu_{k}^j$ and the desired formation $\Theta$ is probabilistically bounded for all $k \geq k_{\text{lim}}$ by:

$$
\P(D_{L_1}(\mu_{k}^j, \Theta) \geq \varepsilon \in \mathbb{N} + \sigma_k) \leq \frac{n_{\text{rec}}}{4m_{k}^2 \varepsilon \in \mathbb{N}} .
$$

(30)

**Proof:** Let $X_{k, i}^j$ denote the Bernoulli random variable, where $X_{k, i}^j = 1$ represents the event that the $j$th agent is actually located in bin $B[i]$ at the $k$th time instant (i.e., $r_{k, i}^j = 1$) and $X_{k, i}^j = 0$ otherwise (i.e., $r_{k, i}^j = 0$). We get from (22) that $\P(X_{k, i}^j = 1) = \tau^j_{k, i}$. Therefore $\E[X_{k, i}^j] \leq 1 - \frac{1}{\Theta}$, where $\E[\cdot]$ and $\Var[\cdot]$ respectively denote the expected value and the variance of the random variable. It follows from Theorem 5...
that $D_{L_1}(x_k^j, \Theta) \leq \sigma_k$ for all $k \geq k_{c,\text{lim}}$. Therefore $|E[X_{k,i}^j] - \Theta[i]| \leq \sigma_k$ for all $k \geq k_{c,\text{lim}}$.

The swarm distribution in bin $B[i]$ at the $k$th time instant is given by $\mu_k^i[i] = \frac{1}{m_k} \sum_{j=1}^{m_k} X_{k,i}^j$. Therefore, $E[\mu_k^i[i]] = \frac{1}{m_k} \sum_{j=1}^{m_k} E[X_{k,i}^j]$. The random variables $X_{k,i}^j, j \in \{1, \ldots, m_k\}$ are negatively correlated because:

$$\text{Cov}(X_{k,i}^j, X_{k,i}^j) = E[X_{k,i}^j X_{k,i}^j] - E[X_{k,i}^j]E[X_{k,i}^j] = \frac{(m_k-2)}{m_k} \frac{(m_k-1)}{m_k} \leq 0,$$

where $n_{k,i}$ is the number of agents in bin $B[i]$ at the $k$th time instant and $inom{m_k}{n_{k,i}}$ represents the Binomial coefficient. Therefore, we get:

$$\text{var}(\mu_k^i[i]) = \frac{1}{m_k^2} \sum_{j=1}^{m_k} \text{var}(X_{k,i}^j) + 2 \sum_{1 \leq j < k \leq m_k} \text{Cov}(X_{k,i}^j, X_{k,i}^j) \leq \frac{1}{4m_k}.$$

It follows from Chebychev’s inequality (cf. [63, Theorem 1.6.4, pp. 25]) that for any $\varepsilon_{\text{bin}}$, the pointwise error probability for each bin is bounded by:

$$\Pr(|\mu_k^i[i] - E[\mu_k^i[i]]| \geq \varepsilon_{\text{bin}}) \leq \frac{1}{4m_k\varepsilon_{\text{bin}}^2}.$$

It follows from the triangle inequality that $|\mu_k^i[i] - E[\mu_k^i[i]]| \geq |\mu_k^i[i] - \Theta[i]| - \sigma_k$, therefore:

$$\Pr(|\mu_k^i[i] - \Theta[i]| \geq \varepsilon_{\text{bin}} + \sigma_k) \leq \frac{1}{4m_k\varepsilon_{\text{bin}}^2}.$$

The bound on $L_1$ distance is obtained using Boole’s inequality:

$$\Pr(\Delta_{L_1}(\mu_k^i, \Theta) \geq \varepsilon_{\text{bin}} + \sigma_k) \leq \sum_{i=1}^{n_{\text{rec}}} \Pr(|\mu_k^i[i] - \Theta[i]| \geq \varepsilon_{\text{bin}} + \sigma_k) \leq \frac{n_{\text{rec}}}{4m_k\varepsilon_{\text{bin}}^2}.$$

The bound on $HD$ follows from (5). Since $\lim_{k \to \infty} x_k^j = \Theta$, we get $\lim_{k \to \infty} E[X_{k,i}^j] = \Theta[i]$ and $\lim_{k \to \infty} \sigma_k = 0$. Setting $\varepsilon_{\text{bin}} = 2\varepsilon_2^2$, we get:

$$\Pr(D_H(\lim_{k \to \infty} \mu_k^i, \Theta) \geq \varepsilon_{\text{des}}) \leq \frac{n_{\text{rec}}}{16m_k\varepsilon_{\text{des}}^2}.$$

The lower bound on the number of agents is given by $n_{\text{rec}} \geq \frac{16m_k\varepsilon_{\text{des}}^2}{\Delta_{\text{conv}}}$. 

Remark 7. It follows from Theorem 6 and the weak law of large numbers [64, pp. 86] that the final swarm distribution $\lim_{k \to \infty} \mu_k^i$ converges in probability to the desired formation $\Theta$ as the number of agents $m_k$ tends to infinity.

Thus, we have proved the convergence of the PSG–IMC algorithm for shape formation. We now discuss its property of robustness and some extensions.

Remark 8. (Robustness of the PSG–IMC Algorithm) The PSG–IMC algorithm satisfies the Markov (memoryless) property because the action of each agent depends only on its present bin location and the current swarm distribution. This property ensures that all the agents re-start their guidance trajectory from their present bin location during every time instant. Thus, the swarm continues to converge to the desired shape even if agents are added to or removed from the swarm, or if some agents have not reached their target bin during the previous time instant.

Moreover, the PSG–IMC algorithm can tolerate estimation errors $\varepsilon_{\text{est}}$ (3) in the feedback of the current swarm distribution $\mu_k^i$. The distance between the feedback gains $\xi_k^i = D_H(\Theta, \mu_k^i)$ and $\xi_k^j = D_H(\Theta, \mu_k^j)$ is bounded by [55]:

$$|\xi_k^i - \xi_k^j| \leq D_H(\mu_k^i, \mu_k^j) \leq \frac{1}{\sqrt{2}} \varepsilon_{\text{est}}.$$

Even though $\xi_k^i$ might differ from $\xi_k^j$ substantially, the resulting Markov matrix $M^i_k$ still has $\Theta$ as its stationary distribution. Therefore the agent’s PMF $x_k^j$ still converges to $\Theta$, and consequently the swarm distribution also converges to $\Theta$. 

Remark 9. (Collision Avoidance in the PSG–IMC Algorithm) The PSG–IMC algorithm can handle inter-agent collision avoidance using line 15 in Method 2. We have not explicitly considered collision avoidance with stationary obstacles because they can be easily handled by the existing framework. If the stationary obstacles are comparable or larger than the bin size, then the bins are designed such that they do not overlap with these obstacles. Furthermore, the motion constraints matrices are designed to prevent transitions that are not allowed due to these obstacles. If the stationary obstacles are significantly smaller than the bins, then they can be handled by the lower-level collision avoidance algorithm as shown in Remark 15 in Appendix. 

Figure 8. This image shows the different thresholds and the corresponding resolution of the desired formation that should be used when $\xi_k^i$ is within those thresholds.

Remark 10. (Multiresolution PSG–IMC Algorithm for Shape Formation) We can take advantage of multiresolution representation of the desired formation in our guidance strategy (see Fig. 6). The key idea is that the agents use an appropriate resolution of the desired formation depending on the feedback gain. For example, as shown in Fig. 8, we select thresholds $\xi_{\text{th},150}, \xi_{\text{th},75}$, and $\xi_{\text{th},25}$ so that the agents use the appropriate resolution of the desired formation ($v_{300}$, $v_{150}$, $v_{75}$ or $v_{25}$ in Fig. 6) if the feedback gain $\xi_k^i$ is within those thresholds. The main advantage of this approach is its computational efficiency.

Remark 11. (Time-Varying Physical Space of the Swarm) The compact physical space over which the swarm is distributed need not be time-invariant in the global reference frame. The local reference frame of the swarm can follow a predefined trajectory in the global reference frame (e.g., an orbit in space or a trajectory in the sea) and the time-varying position of each bin can be computed from this known trajectory. Consequently,
all the algorithms discussed in this paper are also applicable in this scenario.

V. PSG–IMC ALGORITHM FOR AREA EXPLORATION

In this section, we present an extension of the PSG–IMC algorithm for area exploration in which a swarm of distributed agents are driven to match the unknown target distribution of some physical or artificial phenomena (e.g., oil spill). This problem is commonly called goal searching [28].

Definition 11. (Unknown Target Distribution $\Omega$) The unknown target distribution $\Omega$ is a probability (row) vector in $\mathbb{R}^{n_{bin}}$, where each element $\Omega[i]$ represents the target distribution in the corresponding bin $B[i]$. Each agent can measure the target distribution in its present bin.

Each agent independently determines its bin-to-bin trajectory using the PSG–IMC algorithm for area exploration so that the overall swarm converges to this unknown target distribution $\Omega$. The key idea of this algorithm is that the waiting time in a bin is directly proportional to the target distribution in that bin.

Method 3. PSG–IMC Algorithm for Area Exploration

1: Lines 1–2 in Method 2 and given $\tau_c$, $\xi^j$
2: Measure target distribution in present bin $\Omega[i]$
3: if $k - k_0 < \tau_c \Omega[i]$, then
4: Wait in bin $B[i]$
5: else Set $\Theta = \frac{x^j}{n_{bin}}$ and $\xi^j_k = \xi^j$
6: Compute the term $\eta^j_k$, using (33)
7: Lines 9, 11–13, 15 in Method 2. Set $k_0 = k$
8: end if

The pseudo-code for this PSG–IMC algorithm for area exploration is given in Method 3. The $j$th agent first measures the target distribution in its present bin $\Omega[i]$ (line 2). The waiting time in bin $B[i]$ is greater than or equal to $\tau_c \Omega[i]$, where $\tau_c$ is the constant of proportionality.

Assumption 7. (Waiting Time Constant $\tau_c$) Let $\Delta$ denote the time step of the algorithm. Each agent has a-priori information about the common waiting time constant $\tau_c$, which is selected such that $\tau_c > \frac{\Delta}{\min \tau_{\Omega}}$. The actual value of $\tau_c$ plays a crucial role in the convergence analysis.

The agent checks if it has spent enough time instants in bin $B[i]$ (line 3), where $k_0$ is set in line 7. When the algorithm starts ($k = 1$), we set $k_0 = 1$. If the agent has not spent enough time instants in bin $B[i]$, then it continues to wait in bin $B[i]$ (line 4).

If the agent has spent enough time instants in bin $B[i]$, then it sets $\Theta = \frac{x^j}{n_{bin}}$ because it wants to uniformly explore all the bins (line 5). The agent sets the feedback gain $\xi^j_k$ to some positive known constant $\xi^j_k \in (0, 1)$ because the it does not know the target distribution $\Omega$ (line 5). In order to suppress undesirable transitions from bins that are deficient in agents, the agent computes the term $\eta^j_k$ as follows (line 6):

$$\eta^j_k = \exp(-\tau^j k) \frac{\exp(\beta \Omega[i] - \mu^j_k[i])}{\exp(\beta \Omega[i] - \mu^j_k[i])}. \quad (33)$$

Then, the agent computes the transition probabilities $P^j_k[i, 1: n_{bin}]$ using lines 9 and 11 from Method 2, selects the next bin using lines 12–13 from Method 2, and goes to the selected bin using line 15 from Method 2 (line 7). Finally, the agent sets $k_0$ equal to the current time instant $k$ (line 7). We now discuss the convergence analysis of this algorithm.

Theorem 7. (Convergence of IMC for Area Exploration) Let $\Omega \in \mathbb{R}^{n_{bin}}$ be a probability (row) vector. Let $n_{\Omega}$ denote the number of bins with nonzero elements in $\Omega$. According to Method 3, the PMF vector $x^j_k$ converges pointwise to the distribution $\Omega$ irrespective of the initial condition, where

$$D_{L_1}(\hat{\Omega}, \Omega) \leq n_{\Omega} \frac{\Delta}{(n_{bin} - n_{\Omega}) \tau_c} + 1.$$ 

If $\tau_c \gg \Delta$, then $\lim_{k \to \infty} x^j_k = \Omega$ pointwise for all agents.

Proof: Here, all bins are recurrent bins because $\Theta = \frac{x^j}{n_{bin}}$. It follows from Theorem 4 that as $k \to \infty$, an agent is equally likely to transition to any bin $B[i]$. But the waiting time in each bin, under Assumption 7, is given by:

$$\text{Waiting time in bin } B[i] = \begin{cases} \Delta \frac{\tau \Omega[i]}{\Delta} & \text{if } i \in \mathcal{V} \\ \Delta & \text{if } i \in \mathcal{V}^c \end{cases},$$

where $\mathcal{V}$ represents the set of all bins $B[i]$ where $\Omega[i] > 0$, and $\mathcal{V}^c$ represents the set of all bins $B[i]$ where $\Omega[i] = 0$. Note that $|\mathcal{V}| = n_{\Omega}$. Therefore, $\lim_{k \to \infty} x^j_k = \Omega$ pointwise for all agents, where

$$\hat{\Omega}[i] = \begin{cases} \frac{\Delta \tau \Omega[i]}{\Delta \tau \Omega[i]} & \text{if } i \in \mathcal{V} \\ \frac{\Delta \tau \Omega[i]}{\Delta \tau \Omega[i]} & \text{if } i \in \mathcal{V}^c \end{cases}.$$ 

The $L_1$ distance between $\hat{\Omega}$ and $\Omega$ is given by:

$$D_{L_1}(\hat{\Omega}, \Omega) = \sum_{i \in \mathcal{V}} \Delta \frac{\tau \Omega[i]}{\Delta} + \sum_{i \in \mathcal{V}^c} \Delta \frac{\tau \Omega[i]}{\Delta} - \Omega[i] + \sum_{i \in \mathcal{V}} \frac{\Delta \tau \Omega[i]}{\Delta} - \Omega[i] \leq n_{\Omega} \frac{\Delta}{(n_{bin} - n_{\Omega}) \Delta + \tau_c}.$$ 

If $\tau_c \gg \Delta$, then $D_{L_1}(\hat{\Omega}, \Omega) = 0$ and $\lim_{k \to \infty} x^j_k = \Omega$ pointwise for all agents.

The remaining convergence analysis straightforwardly follows that of the previous algorithm given in Section IV-A.

VI. NUMERICAL SIMULATIONS AND EXPERIMENTS

In this section, we present results of numerical simulation and experimentation for the shape formation algorithm in Sections VI-A–VI-C and results of numerical simulations for the area exploration algorithm in Section VI-D.
increases the HD. Therefore, the HD for the HMC algorithm from bins with fewer agents to bins with surplus agents) which HMC algorithm allows undesirable transitions (i.e., transitions (i) In Fig. 9(b), the HD of the HMC algorithm reaches an superior performance of the PSG–IMC algorithm are as times reduction in the cumulative number of transitions in algorithm provides approximately PSG–IMC and HMC-based algorithms are shown in Fig. 9(b). (b) The cumulative results of 10 Monte Carlo simulations are shown. The discontinuity at the 251\textsuperscript{st} time instant is because of the removal of agents from the top half of the formation.

### A. Numerical Simulations for Shape Formation with Fine Spatial Resolution

In this subsection, we show that the PSG–IMC algorithm for shape formation can be used to accomplish multiple complex formation shapes with fine spatial resolutions. At the start of each simulation, a swarm of \(10^5\) agents are uniformly distributed across the physical space. During each time instant, each agent gets the error-free feedback of the current swarm distribution \(\mu_k\). The cost of transition is equal to the \(\ell_1\) distance between bins, therefore it is symmetric. We use the following constants \(\varepsilon_M = 1, \varepsilon_C = 0.1, \tau^j = 10^{-3}, \) and \(\beta^j = 1.8 \times 10^9\).

In the first example, the desired formation \(\Theta\) is that of the Eiffel Tower (\(\Theta_{100}\) in Fig. 6, with 300 \times 300 bins). Each agent is allowed to transition to only those bins that are at most 50 steps away. Starting from a uniform distribution, the agents attain the desired formation in 100 time instants (see Fig. 9(a)). At the 251\textsuperscript{st} time instant, approximately \(3 \times 10^5\) agents are removed from the top half of the formation and the remaining agents reconfigure into the desired formation.

For the HMC-based algorithm, we generate the homogeneous Markov matrix using (10)-(11) and setting \(\beta_k^{1\times} = 1\). The cumulative results of 10 Monte Carlo simulations for the PSG–IMC and HMC-based algorithms are shown in Fig. 9(b). Compared to the HMC-based algorithm, the PSG–IMC algorithm provides approximately 2 times improvement in HD, 16 times reduction in the cumulative number of transitions in 500 time instants, and 16 times reduction in the total cost incurred by all the agents in 500 time instants. The key reasons behind the superior performance of the PSG–IMC algorithm are as follows:

(i) In Fig. 9(b), the HD of the HMC algorithm reaches an equilibrium at 0.115 after approximately 40 time instants. The HMC algorithm allows undesirable transitions (i.e., transitions from bins with fewer agents to bins with surplus agents) which increases the HD. Therefore, the HD for the HMC algorithm reaches an equilibrium because these undesirable transitions reach an equilibrium with the other favorable transitions. Such undesirable transitions are largely avoided in the PSG–IMC algorithm (due to lines 10–11 in Method 2), hence the resulting HD after 250 time instants is 0.055 (i.e., \(\approx 2\) times improvement compared to HMC). The final HD can be further reduced by tuning \(\tau^j\) and \(\beta^j\). But such undesirable transitions prevent both these Markovian approaches from truly achieving zero convergence error.

(ii) In the HMC algorithm, there are \(1.9 \times 10^6\) transitions in the first 40 time instants. This is significantly more than that of the PSG–IMC algorithm (i.e., \(5.6 \times 10^5\) transitions in 250 time instant). In the PSG–IMC algorithm, the number of transitions at each time instant is proportional to the HD. This helps in achieving faster convergence (when HD is large) while avoiding unnecessary transitions (when HD is small). This also ensures that the agents settle down after the desired formation is achieved. Note that the total number of transitions in the HMC algorithm in 250 time instant is extremely large (i.e., \(1.2 \times 10^7\) transitions).

(iii) There are 7–9 agents in each recurrent bin. For the PSG–IMC algorithm, the number of bins with 1–2 excess agents (i.e., 10–20\%) is shown in Fig. 9(b). The number of bins with a large number of excess agents (i.e., 50–100\%) is a small fraction of the total number of bins. Hence this algorithm also avoids traffic jams or bottlenecks. Consequently, the PSG–IMC algorithm achieves a smaller convergence error than the HMC-based algorithm and significantly reduces the number of transitions for achieving and maintaining the desired formation. Moreover, these two key reasons depend on the feedback and, therefore, do not hold true for HMC-based algorithms.

In the next example, the desired formation \(\Theta\) is that of the 3-D Eiffel Tower (see Fig. 10, with \(150 \times 150 \times 150\) bins). Starting from a uniform distribution and no motion constraints, a swarm of \(10^5\) agents achieve the desired formation in a few
time instants. When $1.25 \times 10^4$ agents are removed from the top half of the formation, the remaining agents reconfigure to the desired formation in a few more time instants.

In the next example, the desired formation $\Theta$ in Fig. 11(d), with $325 \times 325$ bins, has multiple disconnected parts. Each agent is allowed to transition to only those bins that are at most 50 steps away. In this case, the recurrent bins are not contiguous, but they satisfy property (3) in Definition 4. A swarm of $10^6$ agents starts from the left-most bin (located at $(1, 163)$) and attains the desired formation in 300 time instants (as shown in Fig. 11(a)–(d)).

In Table II, the computation times using the PSG–IMC and HMC-based algorithms on a desktop computer are shown. The simulation setup for all these runs is exactly the same as shown in Fig. 9. Although both the algorithms scale well with the spatial resolution of the desired distribution and the number of agents in the swarm, the PSG–IMC algorithm performs better because of the smaller number of transitions. Thus, the robustness and scalability properties of the PSG–IMC algorithm for shape formation are evident in these simulation results.

**Table II**

<table>
<thead>
<tr>
<th>Desired Distribution</th>
<th>Number of Agents</th>
<th>PSG–IMC</th>
<th>HMC</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\nu_{25}$</td>
<td>$5000$</td>
<td>15 seconds</td>
<td>23 seconds</td>
</tr>
<tr>
<td>(625 bins)</td>
<td>$10^4$</td>
<td>32 seconds</td>
<td>43 seconds</td>
</tr>
<tr>
<td></td>
<td>$10^6$</td>
<td>5 minutes</td>
<td>6 minutes</td>
</tr>
<tr>
<td></td>
<td>$10^8$</td>
<td>45 minutes</td>
<td>54 minutes</td>
</tr>
<tr>
<td>$\nu_{75}$</td>
<td>$5000$</td>
<td>4.5 minutes</td>
<td>5 minutes</td>
</tr>
<tr>
<td>(5625 bins)</td>
<td>$10^4$</td>
<td>3 minutes</td>
<td>3.5 minutes</td>
</tr>
<tr>
<td></td>
<td>$10^6$</td>
<td>30 minutes</td>
<td>36 minutes</td>
</tr>
<tr>
<td></td>
<td>$10^8$</td>
<td>5 hours</td>
<td>5.9 hours</td>
</tr>
<tr>
<td>$\nu_{150}$</td>
<td>$10^4$</td>
<td>2.0 hours</td>
<td>2.4 hours</td>
</tr>
<tr>
<td>(2.25 $\times$ 10^4 bins)</td>
<td>$10^6$</td>
<td>25 hours</td>
<td>1 day</td>
</tr>
<tr>
<td>$\nu_{100}$</td>
<td>$10^4$</td>
<td>5.3 hours</td>
<td>8 hours</td>
</tr>
<tr>
<td>(9 $\times$ 10^4 bins)</td>
<td>$10^6$</td>
<td>2.5 days</td>
<td>3.3 days</td>
</tr>
</tbody>
</table>

**B. Numerical Simulations for Shape Formation with Coarse Spatial Resolution and Estimation Errors**

The objective of this subsection is to study the effect of estimation errors on the three Eulerian algorithms, namely PSG–IMC, HMC, and the Probabilistic Swarm Guidance using Optimal Transport (PSG–OT) algorithm [36] (see Remark 12 in Appendix). The desired formation $\Theta$ is given by the coarse Eiffel Tower ($\nu_{25}$ in Fig. 6, with $25 \times 25$ bins) because the PSG–OT’s LP (35) cannot be solved for finer resolutions. The simulation setup is similar to that in Section VI-A. A swarm of 5000 agents is used and each agent is allowed to transition to only those bins which are at most 9 steps away.

During each time instant, each agent receives the feedback of the current swarm distribution $\mu^f_k$ with an estimation error $\epsilon_{est}$. The cumulative results of the three algorithms and alternative functions for $\xi_k$ are shown, with an estimation error $\epsilon_{est} = 0.25$.

**Figure 11.** This plot shows the swarm distribution at different time instants, where the swarm attains the desired formation shape with multiple disconnected parts. See the supplementary video (SV3).

**Figure 10.** The swarm attains the 3-D shape of the Eiffel Tower. When agents are removed from the top half of the formation, the remaining agents reconfigure to the desired formation.

**Figure 12.** (a) The estimation error is varied from 0.0 to 0.5. The performance of the three algorithms, along with 1σ error-bars, are shown for the true HD $D_H(\Theta, \mu^{250}_k)$ between the actual swarm distribution after 250 time instants and the desired formation, the cumulative number of transitions in 250 time instants, and the total cost incurred by all the agents in 250 time instants. (b) The cumulative results of the three algorithms and alternative functions for $\xi_k$ are shown, with an estimation error $\epsilon_{est} = 0.25$. On the other hand, the PSG–IMC algorithm works reliably well for all estimation errors and performs much better than the other two algorithms. Thus, the PSG–IMC algorithm can tolerate large estimation errors.
in the feedback of the current swarm distribution.

The cumulative results for the three algorithms are shown in Fig. 12(b), where the estimation error $\epsilon_{est}$ is equal to 0.25. Compared to the HMC and PSG–OT algorithms, the PSG–IMC algorithm achieves a smaller convergence error with fewer transitions. The results of a few alternative functions for $\xi_k$ are also shown in Fig. 12(b) (see Remark 4). The two functions $f_1(\xi_k^j) = \tanh(\pi\xi_k^j)$ and $f_2(\xi_k^j) = \sin^{-1}(1 - \xi_k^j)$ are always larger than $\xi_k^j$. The sigmoid function $f_3(\xi_k^j) = (\xi_k^j + 0.1 \sin(2\pi\xi_k^j))$ is larger than $\xi_k^j$ when $\xi_k^j < 0.5$. Fig. 12(b) shows that the rate of convergence increases with these functions, but there is also a corresponding increase in the number of transitions. The collision-free motion of the agents, where the estimation error $\epsilon_{est} = 0.25$ and the agents use the Voronoi-based lower-level algorithm in Remark 15, is shown in the supplementary video (SV4).

![Supplementary Video](sv4)

Figure 12(b) shows that the rate of convergence increases with these functions, but there is also a corresponding increase in the number of transitions.

C. Experimental Results for Shape Formation to Demonstrate Real-Time Execution

In this subsection, we show that the PSG–IMC algorithm and the lower-level algorithm in Remark 15 can be executed in real-time using quadrotors. The experimental setup is shown in Fig. 13 and described in [7], [65]. A $3 \times 3$ grid is placed on the ground where the quadrotor experiments are performed. The quadrotors are tracked using the Vicon motion capture cameras and the Vicon server fuses the data from these cameras to estimate the position of these quadrotors to $1 - 2$ cm accuracy. A desktop computer receives the position information from the Vicon server and executes the PSG–IMC algorithm for each agent in a virtually distributed manner, i.e., each quadrotor’s computations are performed by an independent thread on the desktop so that the experimental setup mimics a distributed system. The trajectories computed by each quadrotor’s thread are then communicated to that quadrotor. Finally, each quadrotor tracks its computed trajectory using an onboard nonlinear control law [7], [65].

We first present nine different experiments using three or five quadrotors. The desired formation shape for these experiments are shown in Fig. 14. In these experiments, the time step of the PSG–IMC algorithm is 9 seconds and the time step for the lower-level guidance algorithm in Remark 15 is 3 seconds. As shown in the supplementary video (SV5), the Crazyflie quadrotors first take off from the ground and climb to 1 m altitude. Thereafter, the PSG–IMC algorithm is switched on, and the quadrotors achieve the desired formation shape within a few time instants. The quadrotor then land inside their selected bins. Note that these exists some parallax error in the video (SV5) because the grid is marked on the ground, the quadrotors are flying at 1 m altitude, and the camera is located directly above the central square at 5 m height. This parallax error vanishes when the quadrotors land and the desired formation shape is clearly visible in the end of video (SV5). The quadrotors experience measurement errors, actuator errors, and inter-quadrotor aerodynamic coupling due
three quadrotors reach the desired formation highlighted in green in Fig. 14. These repetitive experiments show that the quadrotors select different bins during different runs due to the probabilistic nature of the PSG–IMC algorithm.

D. Numerical Simulations for Area Exploration

In this numerical example, a swarm of $10^5$ agents use the PSG–IMC algorithm for area exploration to attain the unknown target distribution. The physical space $[0, 1] \times [0, 1]$ is partitioned into $100 \times 100$ bins and the time step $\Delta = 0.1$ sec. The unknown target distribution $\Omega_1$ for the first 100 sec is given by the PMF representation of the multivariate normal distribution $\mathcal{N}([0.5, 0.5], \begin{bmatrix} 0.1 & 0.3 \\ 0.3 & 1.0 \end{bmatrix})$, as shown in the background contour plots in Fig. 16(a). Similarly, the unknown target distribution $\Omega_2$ for the next 100 sec is given by $\mathcal{N}([0.5, 0.5], \begin{bmatrix} -0.3 & 1.0 \\ 1.0 & -0.3 \end{bmatrix})$. Here we use the constants $\tau^j = 2.5 \times 10^{-3}$, and $\beta^j = 200$ in (33).

The cumulative results of 10 Monte Carlo simulations for different values of $\xi^j$ are shown in Fig. 16(b). A results for the HMC-based area exploration algorithm are also shown in Fig. 16(b). Compared to the HMC-based algorithm, the PSG–IMC algorithm provides approximately 1.5 times improvement in HD, 6 times reduction in the cumulative number of transitions and the total cost incurred by the agents in 200 sec.

VII. Conclusion

In this paper, we have presented a new distributed control algorithm for large-scale swarms by using the Eulerian framework. Our highly scalable, robust, and versatile PSG–IMC algorithm ensures that a swarm converges to the desired formation or the unknown target distribution.

In our PSG–IMC algorithm, time-inhomogeneous Markov matrices with a desired stationary distribution are systematically constructed using the Hellinger-distance-based feedback gain, which incorporates feedback from the current swarm.
distribution. These Markov matrices satisfy suitable motion constraints, minimize the expected cost of transitions at each time instant, and circumvent transitions from bins that are deficient in the number of agents. We have also presented a rigorous convergence analysis of the PSG–IMC algorithm. To our knowledge, this PSG–IMC algorithm we have presented in this paper is the first path planning strategy that leverages the idea of constructing IMC, in real-time, based on feedback of the current swarm distribution.

We have carried out numerical simulations which show that the PSG–IMC algorithm achieves 6 – 16 times reduction in total cost of transitions and 1.5 – 2 times reduction in HD, as compared to existing HMC-based algorithms for shape formation and area exploration applications. This is because the PSG–IMC algorithm avoids undesirable transitions, and the number of transitions at each time instant is proportional to the HD. In the presence of estimation errors, our PSG–IMC algorithm also outperforms the optimal transport-based algorithm, because the PMF of the predicted position of each agent converges to the desired formation regardless of estimation errors. We have demonstrated the robustness and computational benefits of the PSG–IMC algorithm using hardware experiments with multiple quadrotors, where a Voronoi-based lower-level guidance and control algorithm is used. This also provides an avenue for future research in tightly integrating a lower-level guidance and control algorithm with our PSG–IMC algorithm.

The PSG–IMC algorithm presented in this paper can also solve other cooperative control tasks, such as surveillance, task allocation, and coverage, since such problems can also be cast as shape formation or area exploration problems. This is because the idea of constructing IMC, in real-time, based on feedback of the current swarm distribution.

Theorem 8. [46, pp. 150] If (i) the forward matrix product $U_{T,r}$ is primitive and (ii) there exists $\gamma$ (independent of $k$) such that:

$$0 < \gamma \leq \min_{i, \ell} P_k[i, \ell],$$

where $\min^+$ refers to the minimum of the positive elements, then the asymptotic homogeneity of $P_k$ is necessary and sufficient for strong ergodicity of $U_{T,r}$.

Theorem 9. [46, pp. 149] Let $e_k$ be the unique stationary distribution vector of the matrix $P_k$ (i.e., $e_k P_k = e_k$). If (i) all $P_k$, $k \geq 1$ are irreducible and (ii) there exists $\gamma$ (independent of $k$) such that (34) is satisfied, then asymptotic homogeneity of $P_k$ (with respect to $d$) is equivalent to $\lim_{k \to \infty} e_k = e$, where $e$ is a limit vector. Moreover, $d = e$.

Corollary 3. [46, pp. 150] Under the prior conditions (i) and (ii) of Theorem 9 and if (iii) $U_{T,r}$ is strongly ergodic with unique limit vector $v$, then $v = e$.

Theorem 10. The matrix $U^j_{k,k+n_{rec}-1}$ is a positive matrix.

Proof: By Contradiction. Assume that $U^j_{k,k+n_{rec}-1}[i, \ell] = 0$ for some $i \neq \ell$. Then $U^j_{k,r}[i, \ell] = 0$ for all $r \leq k + n_{rec} - 2$ because $U^j_{k,k+n_{rec}-1}[i, \ell] \geq U^j_{k,r}[i, \ell] \left( \prod_{q=r}^{k+n_{rec}-2} P^j_{q,sub}[\ell, \ell] \right)$. Since the matrix $P^j_{k+n_{rec}-2,sub}$ is irreducible, there exists $s_1 \in \{1, \ldots, n_{rec}\} \setminus \{i, \ell\}$ such that $P^j_{k+n_{rec}-2,sub}[s_1, \ell] > 0$. Since $U^j_{k,k+n_{rec}-1}[i, \ell] \geq U^j_{k,k+n_{rec}-2}[i, s_1] P^j_{k+n_{rec}-2,sub}[s_1, \ell]$, consequently $U^j_{k,r}[i, s_1] = 0$ for all $r \leq k + n_{rec} - 2$. Similarly, there exists $s_2 \in \{1, \ldots, n_{rec}\} \setminus \{i, s_1\}$ such that either $P^j_{k+n_{rec}-3,sub}[s_2, \ell] > 0$ or $P^j_{k+n_{rec}-3,sub}[s_2, s_1] > 0$. Therefore $U^j_{k,r}[i, s_2] = 0$ for all $r \leq k + n_{rec} - 3$. Continuing this argument till the $k^{th}$ time instant, we see that if $U^j_{k,k+n_{rec}-1}[i, \ell] = 0$, then $P^j_{k,sub}[i, s] = 0$ for all $s \in \{1, \ldots, n_{rec}\} \setminus \{i\}$. But this is a contradiction since $P^j_{k,sub}$ is irreducible.

Remark 12. [36] (PSG–OT Algorithm) The cost function $C_k$ from Definition 5 is first modified to capture motion constraints, i.e., $C_k[i, \ell] = C_{\ell}[i, \ell]$ if $A^{\ell}_i[i, \ell] = 1$ and $C_k[i, \ell] = C_{\max}$ otherwise, where $C_{\max} \gg C_{\ell}[i, \ell]$ for all $i, \ell$. The optimum transference plan $\Gamma_k^i \in \mathbb{R}_{\max} \times n_{rec}$ is found using the following LP:

$$\min_{\Gamma^i_k[i, \ell] \geq 0, \forall i, \ell} \sum_{i=1}^{n_{bin}} \sum_{\ell=1}^{n_{bin}} C_k[i, \ell] \Gamma^i_k[i, \ell],$$

subject to (i) $\sum_{\ell=1}^{n_{bin}} \Gamma^i_k[i, \ell] = \mu^i_k[i, \ell], \forall i$, (ii) $\sum_{i=1}^{n_{bin}} \Gamma^i_k[i, \ell] = \Theta[\ell], \forall \ell$.

Note that the feedback of the current swarm distribution $\mu^i_k$ directly enters (35), hence $\Gamma^i_k$ is sensitive to estimation errors.
The matrix $\Gamma_k$ is not a Markov matrix because it is not row stochastic.

Remark 13. [51] (Distributed Estimation of $\mu_k^j$) Let the probability vector $\hat{\mu}_{k,v}^j \in \mathbb{R}^{m_v}$ represent the $j^{th}$ agent’s estimate of the current swarm distribution during the $v^{th}$ consensus loop at the $k^{th}$ time step. During each consensus loop, the agents recursively combine their local estimates with their neighboring agents as follows:

$$
\hat{\mu}_{k,v+1}^j = \sum_{\ell \in \mathcal{J}_k^j} a_{k,\ell} \hat{\mu}_{k,v}^\ell, \quad \forall v \in \mathbb{N},
$$

where $\mathcal{J}_k^j$ is the set of inclusive neighbors of the $j^{th}$ agent, $\sum_{\ell \in \mathcal{J}_k^j} a_{k,\ell} = a_k$, is irreducible. Distributed algorithms in [66], [67] can ensure that the matrix $A_k$ is balanced. Then each agents local estimate $\hat{\mu}_{k,v}^j$ globally exponentially converges to $\mu_k^j$ pointwise with a rate $r_k$ equal to the second-largest singular value of $A_k$ (i.e., $\sigma_{m_k-1}(A_k)$). For some $\varepsilon_{\text{cons}} > 0$, if the number of consensus loops within each consensus stage $n_{\text{loop}} \geq \frac{\ln(1/\varepsilon_{\text{cons}}/2m)}{\ln(\sigma_{m_k-1}(A_k))}$, then the convergence error is bounded by $\Delta \mu_k^j = \sum_{\ell \in \mathcal{J}_k^j} a_{k,\ell} (\hat{\mu}_{k,v}^\ell - \mu_k^j) \leq \varepsilon_{\text{cons}}$. The $j^{th}$ agent’s estimate of the current swarm distribution at the $k^{th}$ time step instant is given by $\mu_k^j = \hat{\mu}_{k,n_{\text{loop}}}^j$.

Remark 14. [34] (Inverse Transform Sampling) This a well-known sampling technique for generating samples at random from a given PMF over the set of bins. The key steps are: (1) Generate a uniform random number $z$ in the interval [0, 1]. (2) Represent the PMF as a cumulative distribution function (CDF). (3) Find bin $B[i]$ such that CDF of bins up to (but not including) bin $B[i]$ is less than $z$; and the CDF of bin $B[i]$ is greater than or equal to $z$. Then, the bin $B[i]$ is selected as the sample.

Figure 17. The motion of six agents in two bins (left and right bin) using the Voronoi-based collision-free trajectory generation algorithm are shown. The red agent goes from the left bin to the right bin. The remaining blue agents stay in their present bin. The Voronoi sets of all the agents along with their trajectories (denoted using arrows) are also shown.

Remark 15. [36] (Voronoi-based Collision-free Motion to Selected Bin) As shown in Fig. 17, the agents first generate their Voronoi partitions by communicating with their neighboring agents and considering nearby stationary obstacles. The agents that need to transition to another bin move to the location in their Voronoi partition that is closest to their target bin, while leaving buffer distance for collision avoidance. The agents that want to remain in their present bin move to the centroid of the region where their Voronoi partition overlaps with their present bin. This results in a collision-free trajectory for each agent.

REFERENCES


